An Analysis of the "Kinematic Part" of Einstein's Paper: *On the Electrodynamics of Moving Bodies*

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Introduction

Zur Electrodynamik bewegter Körper (On the Electrodynamics of Moving Bodies) was published in Vol. 17 of the Annalen der Physik in 1905. Two other papers by Albert Einstein appeared in Vol. 17: On the Motion of Small Particles Suspended in Liquids at Rest Required by the Molecular-Kinetic Theory of Heat and On a Heuristic Point of View Concerning the Production and Transformation of Light. Arthur I. Miller states, in his book Albert Einstein's Special Theory of Relativity, "As far as we know the editorial policy of the Annalen was that an author's initial contributions were scrutinized by either the editor or a member of the Curatorium; subsequent papers may have been published with no refereeing. Einstein's having appeared in print in the Annalen five times by 1905, his relativity paper was probably accepted on receipt."[1]

Arthur I. Miller also includes this incident in his description of the initial reception of Albert Einstein's paper, "In the fall of 1907 the relativity paper was rejected by the University of Bern as his Habilitationsschrift. One experimentalist wrote, 'I cannot at all understand what you have written.'"[2] As A. I. Miller notes, in the endnote that accompanies the previous quotation, this assessment was by a professor of experimental physics Aimé Forster.

One of the obstacles that may have hindered Aimé Forster's appreciation the relativity paper is addressed by A. I. Miller, "the first part of the special relativity paper is, in fact, nothing less than an epistemological analysis of the nature of space and time."[3] With this in mind, it seems reasonable to try to gain a clear understanding of Einstein's philosophy of science.

It is difficult to address the significance of Einstein's many comments on the general principles that govern the field of knowledge known as physics. He certainly did not dismiss the notion that the empirical testability of a theory was a crucial criterion for judging a theory's validity. However, it was also crucial to Einstein that the premises of a theory have a naturalness and logical simplicity. This is what Miller refers to as Einstein's idea of the "inner perfection"[4] of a theory. In the endnotes that accompany the "Introduction" to his book, Miller sites a well-known statement by Einstein, which he wrote forty-five years after his completion of the special relativity paper, "in Reply to Criticisms (1949) he [Einstein] described, 'concepts and theories as free inventions of the human spirit (not logically derivable from what is empirically given).'"[5] Statements such as the one above, which seem to refer to the inner perfection of a theory are balanced by statements such as, "The first point is obvious: the theory must not contradict empirical facts. However evident this demand may in the first place appear, its application turns out to be quite delicate."[6] Miller's conclusion is that the essence of Einstein's scientific method was inarticulable.

If we turn our attention to Einstein's book Relativity: The Special and the General Theory, we can examine Einstein's "inarticulable" scientific method in operation. In chapter eight, which is entitled "On the Idea of Time in Physics," he provides us with a definition for determining the simultaneity of distant events. Einstein's definition is written in the form of a dialogue between himself and the reader.

> After thinking the matter over for some time you then offer the following suggestion with which to test simultaneity. By measuring along the rails, the connecting line AB should be measured up and an observer placed at the mid-point M of the distance AB. This observer should be supplied with an arrangement (e.g., two mirrors inclined at 90°) which allows him visually to observe both places A and B at the same time. If the observer perceives the two flashes of lightning at the same time, then they are simultaneous.
> I am very pleased with this suggestion, but for all that I cannot regard the matter as quite settled, because I feel constrained to raise the following objection: "Your definition would certainly be right, if only I knew that the light by means of which the observer at M perceives the lightning flashes travels along the length A ↣ M with the same velocity as along the length B ↣ M. But an examination of this supposition would only be possible if we already had at our disposal the means of measuring time. It would thus appear as though we were moving here in a logical circle."[7]

A minor error in terminology occurs when Einstein expresses his desire to be certain that the velocity of light along the length A ↣ M is the same as the velocity of light along the length B ↣ M. The velocity of light is a vector. A vector has both magnitude and direction. The light beams are traveling in opposite directions so they cannot have the same velocity. However, the light beams can have the same magnitude. If the light-beam vectors have the same magnitude, it would mean their speeds are equivalent, which is the point Einstein is trying to make.

There is another error that is more serious. The definition does not take into consideration that the earth is in motion and further that the motions of the light beams are independent of the earth's motion. In other words, the light beams are not carried along by the earth's motion. All the earthbound objects we commonly observe in motion such cars, planes and trains are carried along by the earth's motion. Also, all the objects we commonly observe as being at rest such as buildings, bridges and telephone poles are carried along by the earth's motion. Thus, the observer standing still at the midpoint M of the length AB is being carried along by the earth's motion. Since the endpoints of the length AB are also being carried along by the earth's motion, the observer at the midpoint M is at rest relative to the endpoints of the length AB. Thus, the observer at the midpoint M maintains a constant distance from the endpoints. This is not the case with the light beams that originate from either endpoint. The observer at the midpoint M is rushing toward one light beam and away from the other light beam, although he seems to be standing still. This is because the earth is in motion and the motion of the light beams are independent of the earth's motion. Of course, both light beams are traveling toward the observer at the midpoint M at the speed of light.

Einstein's definition of a test to determine the simultaneity of distant events provides an example of the naturalness and the logical simplicity that he believed should characterize the premises of a theory. The naturalness of his definition resides in the fact that it agrees with our observations of the everyday world. For example, let an observer stand at the midpoint of a smooth and level stretch of two-lane highway 60 miles in length. Also, station an automobile at either end of this 60-mile length of highway, and let the automobiles start traveling at a given time with a constant speed of 30 mph. If the two automobiles pass by the observer standing at the midpoint at the same instant, the two automobiles began their journey at the same instant. The logical simplicity of Einstein's definition resides in minimal number of measurements required to assess the simultaneity of distant events and the straightforwardness of the observations required.

We can now return to Einstein's dialogue with the reader–picking up where we left off.

> After further consideration you cast a somewhat disdainful glance at me–and rightly so–and you declare: "I maintain my previous definition nevertheless, because in reality it assumes nothing about light. There is only one demand to be made of the definition of simultaneity, namely, that in every real case it must supply us with an empirical decision as to whether or not the conception that has to be defined is fulfilled. That my definition satisfies this demand is indisputable. That light requires the same time to traverse the path A ↣ M as for the path B ↣ M is in reality neither a supposition nor a hypothesis about the physical nature of light, but a stipulation which I can make of my own free will in order to arrive at a definition of simultaneity."[8]

Einstein overreaches with his statement, "There is only one demand to be made of the definition of simultaneity, namely, that in every real case it must supply us with an empirical decision as to whether or not the conception that has to be defined is fulfilled." If instead of stationing the observer at the midpoint M, we stationed the observer at a point one-third of the way from A to B, this observer could still provide us with an empirical decision in every real case, yet no one would conclude that this definition of distant simultaneous events fulfilled the demands required of an accurate definition. Supplying an empirical decision in every real case is not sufficient for an accurate definition of distant simultaneous events. Einstein again overreaches with his statement, “That light requires the same time to traverse the path A ↣ M as for the path B ↣ M is in reality neither a supposition nor a hypothesis about the physical nature of light, but a stipulation which I can make of my own free will in order to arrive at a definition of simultaneity.” The dilemma is that Einstein’s definition of simultaneity requires that distant events that do not occur at the same moment must be regarded as simultaneous. Einstein’s definition of distant simultaneous events is an example of “concepts and theories as free inventions of the human spirit (not logically derivable from what is empirically given).”[9]

In his paper On the Electrodynamics of Moving Bodies, Einstein employs the notion that on a moving body the time it takes a light beam to travel from A ↣ M is different from the time it takes a light beam to travel from B ↣ M. The reason Einstein gives for this phenomenon is the earth’s motion and the independence of light beams from the earth’s motion. He uses this phenomenon to demonstrate the relative simultaneity of distant events. For observers in a system at rest (the earth is not a system at rest), the time it takes a light beam to travel from A ↣ M is equal to the time it takes a light beam to travel from B ↣ M. For observers in a moving system, the time it takes a light beam to travel from A ↣ M is not equal to the time it takes a light beam to travel from B ↣ M.

In his paper, Einstein deals with the question whether a light beam travels with the same speed regardless of the direction in which it is traveling by assuming, as a postulate of his theory, that the speed of light is constant.

Once Einstein has used the fact that on a moving body the time it takes a light beam to travel from A ↣ M is not equal to the time it takes a light beam to travel from B ↣ M, in order to demonstrate the relativity of the simultaneity of distant events, he alters or altogether rejects this notion. He uses a complex mathematical equation in an attempt to demonstrate that on a moving object (or in a moving system) the time it takes a light beam to travel from A ↣ M is equal to the time it takes a light beam to travel from B ↣ M.

If the first part of the special relativity paper is an epistemological examination of the nature of space and time, it is an unfortunate omission that Einstein never addresses why distant clocks must by synchronized solely by beams of light. Light beams have many attributes that are useful in the synchronization distant clocks, but they also have attributes that would not be useful. Perhaps, a discussion of various other ways of synchronizing distant clocks such as mechanical means and the use of sound waves would have clarified Einstein's position. If each method of synchronizing distant clocks has its advantages and disadvantages, perhaps, a method of synchronization that employed both sound waves and light beams, for instance, could overcome by combination any disadvantage possessed by a method dependent on a single means of synchronization.

Christopher Jon Bjerknes begins his book Anticipations of Einstein: In the General Theory of Relativity with a citation from Anthony Berkeley Cox that first appeared in Scribner's Magazine, Volume 88, (July–December 1930). In Charles L. Poor's article for Scribner's Magazine entitled "What Einstein Really Did," he gives us this statement by A. B. Cox, "Artistic proof is, like artistic anything else, simply a matter of selection. If you know what to put in and what to leave out you can prove anything you like, quite conclusively."[10] With that said, there is only one remaining point to cover in this introduction.

The notion that the first part of Einstein's paper, the "Kinematic Part," is incorrect is opposed to the established views on the subject. Therefore, any counter claim must provide a detailed and concise examination of every equation and every significant statement that appears in the first part of Einstein's paper. Also, any counter claim must include an examination of all the intermediate steps between the various equations. Although there are many mathematical equations in each of the following five parts of this paper, they are not difficult to comprehend because they are explained in detail.

Part One

An Analysis of Section 1. Definition of Simultaneity

and Section 2. On the Relativity of Lengths and Times

If we examine a number of passages from Albert Einstein's seminal paper, On the Electrodynamics of Moving Bodies, the paper's inconsistencies and ambiguities will become apparent. These ambiguities and inconsistencies amount to errors that invalidate his claims regarding the concept referred to as the simultaneity of distant events. Einstein claims the simultaneity of distant events cannot have an absolute meaning and that instead its meaning must be relative to a particular coordinate system.

On the Electrodynamics of Moving Bodies is a remarkable accomplishment that has had an enormous impact on the development of modern physics. John Stachel, the editor of Einstein' s Miraculous Year: Five Papers That Changed the Face of Physics, provides a concise description of Einstein's paper and its significance.

> Einstein was the first physicist to formulate clearly the new kinematical foundation for all of physics inherent in Lorentz's electron theory. This kinematics emerged in 1905 from his critical examination of the physical significance of the concepts of spatial and temporal intervals. The examination, based on a careful definition of the simultaneity of distant events, showed that the concept of a universal or absolute time, on which Newtonian kinematics is based, has to be abandoned; and that the Galilean transformations between the coordinates of two inertial frames of reference has to be replaced by a set of spatial and temporal transformations that agree formally with a set that Lorentz had introduced earlier with a quite different interpretation. Through the interpretation of these transformations as elements of a space-time symmetry group corresponding to the new kinematics, the special theory of relativity (as it later came to be called) provided physicists with a powerful guide in the search for new dynamical theories of fields and particles and gradually led to a deeper appreciation of the role of symmetry criteria in physics.[11]

The amount of care (to borrow John Stachel's terminology) Einstein employed in his formulation of the definition of distant simultaneous events is unclear. Certainly, in his thought experiment(s) Einstein did not describe the method by which the clocks in a moving system are synchronized with the clocks in a system at rest. This oversight is significant because testing the synchronization of distant clocks, which are at relative rest in a moving system, is central to Einstein's demonstration of the relativity of simultaneity. In Einstein's crucial thought experiment two different sets of observers must test distant clocks in a moving system for synchronization. One set of observers is co-moving with the moving system, and the other set of observers is at rest in the rest system. In Einstein's thought experiments, a moving rod represents the moving system. He merely informs us that the clocks at either end of the rod are synchronized with the clocks in the rest system.

> Further, we imagine the two ends (A and B) of the rod equipped with clocks that are synchronous with the clocks of the rest system, i.e., whose readings always correspond to the "time of the system at rest" at the locations the clocks happen to occupy; hence, these clocks are "synchronous in the rest system."[12]

Einstein does not tell us the method by which the clocks in the moving system are synchronized with the clocks in the rest system. Furthermore, he never tells us if the rest system is in a state of absolute rest or if it is merely labeled the rest system for convenience or for some deeper reason. If Einstein were to tell us the method by which the clocks in the moving system were synchronized with the clocks in the rest system, this would spell out whether the rest system was in motion or in a state of absolute rest.

It is interesting that John Stachel refers to Einstein's definition of the simultaneity of distant events as "careful." In Einstein's book Relativity: The Special and the General Theory, the definition of simultaneity is referred to as "most natural."[13] The definition of the simultaneity of distant events is not referred to as scientifically rigorous in either instance, which is unexpected for the central tenet of a theory that formed "the new kinematical foundation for all of physics."[14]

In order to clarify Einstein's ideas, we will analyze ten passages from his paper. The following ten passages are taken either from the introductory paragraphs of Einstein's paper or from the first part of his paper, which is entitled "A. Kinematic Part." In the first part of his paper, he formulates the thought experiments that overturn the concept of absolute time. The second part of his paper is entitled "B. Electrodynamic Part." In this part, John Stachel states, "he applied his kinematical results to the solution of a number of problems in the optics and electrodynamics of moving bodies."[15]

The passage below is taken from Einstein's introductory paragraphs. There are two points that should be noted. First, near the beginning of the passage, there is an oblique reference to the Michelson-Morley experiment in the phrase, "unsuccessful attempts to detect a motion of the earth relative to the 'light medium.'"[16] Secondly, the last sentence of the passage is consequential, "The introduction of a 'light ether' will prove superfluous, inasmuch as the view developed here will not require a 'space at absolute rest' endowed with special properties where electro-magnetic processes are taking place."[17] Does Einstein mean that his analysis does not require a 'space at absolute rest' or that his analysis does not require a 'space at absolute rest' endowed with special properties? Only a few paragraphs later Einstein introduces the "rest system."[18] The "rest system" would seem to be a system at absolute rest except for the fact that Einstein is compelled to establish the definition of a common time for points A and B of the rest system "by definition," [19] as opposed to by utilizing the physical properties of the system. These two points will be expanded upon in the following analysis.

> It is well known that Maxwell's electrodynamics—as usually understood at present—when applied to moving bodies, leads to asymmetries that do not seem to be inherent in the phenomena. . ..
> Examples of this sort, together with the unsuccessful attempts to detect a motion of the earth relative to the "light medium," lead to the conjecture that not only the phenomena of mechanics but also those of electro-dynamics have no properties that correspond to the concept of absolute rest. Rather, the same laws of electrodynamics and optics will be valid for all coordinate systems in which the equations of mechanics hold, as has already been shown for quantities of the first order. We shall raise this conjecture (whose content will hereafter be called "the principle of relativity") to the status of a postulate and shall also introduce another postulate, which is only seemingly incompatible with it, namely that light always propagates in empty space with a definite velocity V that is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent electrodynamics of moving bodies based on Maxwell's theory for bodies at rest. The introduction of a "light ether" will prove to be superfluous, inasmuch as the view developed here will not require a "space at absolute rest" endowed with special properties where electro-magnetic processes are taking place.[20]

It should be noted that Einstein employs the term V for the velocity of light instead of the more familiar term c. In Einstein's book Relativity: The Special and the General Theory the term c is employed to represent the velocity of light in empty space, and also there are translations of his 1905 paper that employ the more familiar term c instead of V, such as Arthur I. Miller's translation, which appears in the Appendix of his book, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911).

Einstein's oblique reference to the Michelson-Morley experiment has caught the attention of many commentators. Though most of them have failed to indicate how ill-suited the Michelson-Morley experiment is to make a determination of the absolute motion of the earth with regard to the light ether. The light ether can be conceived of in two ways. First, the light ether can be conceived of as a medium for the transmission of light. In this instance, the light ether is in a state of absolute rest. When the earth moves through this medium, an apparent ether wind is produced that can affect a beam of light. The apparent ether wind is similar to the wind you feel when you ride a bicycle on a windless day. You feel and see the effects of this localized wind, but it is only an apparent wind because when the bicycle stops the wind ceases. Secondly, the light ether can be conceived of as empty space, which can include an observer in a state of absolute rest. In this conception of the light ether, the ether has no influence on a beam of light. Whether the first or second conception of the light ether is employed, the Michelson-Morley experiment is ill-equipped to detect either its influence on a beam of light in the first instance or the influence of the earth's motion on a beam of light in the second instance.

If we view the light ether as a medium for light transmission, the motion of the earth through the ether will influence the path of a light beam in the same way the current of a river will affect a swimmer. The mathematical underpinnings of the Michelson-Morley experiment require that one beam of light that has been split in half by the experimental apparatus must form two rays of light that are perpendicular to each other. The experimental apparatus does produce two rays of light, but the flaw is that the two rays of light are never perpendicular to one another (if we view the ether as a medium for the transmission of light), and no amount of alteration of the experimental apparatus can change this.

If we view the light ether as empty space in which an observer in a state of absolute rest can be located, the empty space will have no influence on a beam of light, but the motion of the earth will have an influence. Under these conditions, the experimental apparatus can produce two light rays that are perpendicular to each other. The flaw in this instance is that the light ray returning along the arm of the experimental apparatus, which is perpendicular to the motion of the earth, will fail to coincide with the ray of light returning along the arm of the apparatus parallel to the motion of the earth. The experimental apparatus will move forward with velocity v, which is the velocity of the earth. The light ray perpendicular to the motion of the earth will not move forward with velocity v so it will lag behind the experimental apparatus. The light ray will, of course, travel with the velocity c in the direction perpendicular to the motion of the earth.

Now, we should return our attention to Einstein's thought experiments in which he will analyze distant simultaneous events. The following five passages are taken from the first section of "A. Kinematic Part," which is entitled, "1. Definition of Simultaneity." The five passages discussed below appear in the same order of occurrence as they appear in the original text.

> Consider a coordinate system in which Newton's mechanical equations are valid. To distinguish this system verbally from those to be introduced later, and to make our presentation more precise, we will call it the "rest system."[21]

There are several reasons why Einstein could have enclosed the phrase "rest system" in quotation marks. The quotation marks may have been used to emphasize the term, to denote the term as a label, or to suggest doubt or skepticism. Einstein could have used the quotation marks to accomplish all three of the tasks. Einstein must be using the term "rest system" as a label since he states, "To distinguish this system verbally from those to be introduced later, and to make our presentation more precise, we will call it the "'rest system.'"[22] However, as the following analysis will show, he is also using the quotation marks to suggest doubt. He wants to cast doubt on the notion that the concept of a rest system and the concept of a moving system are distinct from each another.

Is Einstein's "rest system" at rest relative to some other system that is in motion? For example, trees, buildings and telephone poles are all systems that are at rest relative to the motion of the earth. Is the "rest system" in a state of absolute rest? Two statements from the second introductory paragraph suggest otherwise, since Einstein tells us, "the phenomena . . . of electrodynamics have no properties that correspond to the concept of absolute rest . . .. the view developed here will not require a 'space at absolute rest' endowed with special properties where electromagnetic processes are taking place."[23]

However, Einstein is inconsistent. As he continues his description of the "rest system," the system he describes must be a system in a state of absolute rest. This is so because his mathematical description of a light beam's round-trip journey on a moving body is different from his mathematical description of a light beam's round-trip journey on a body at rest in the "rest system." The mathematical description of a light beam's round-trip journey from point A to point B and back to point A on a moving body is $t_B - t_A = r_{AB}/(V - v)$ and $t'_A - t_B = r_{AB}/(V + v)$, where r_{AB} denotes the distance from point A to point B. Also, the moving body is moving in the direction that makes point B the lead point of the system. If these two formulas represent a light beam's round-trip journey on a moving body, it is reasonable that the formula that represents a light beam's round-trip journey in the "rest system," $t_B - t_A = t'_A - t_B$, should represent a system in a state of absolute rest. Since points A and B are not in motion in the "rest system" the distance a light beam must travel to go from point A to point B is the same distance the light beam must travel to go from point B to point A and hence the duration of each leg of the round-trip journey is the same. This is not so for points A and B on a moving body. When the light beam emanates from point A, point B is rushing away from the oncoming light beam. When the reflected light beam, emanating from point B, is returning to point A, point A is rushing toward the oncoming light beam. Thus, the journey from point A to point B is of greater duration than the return journey from point B to point A.

Before we delve any deeper into the nature of the "rest system," we should explore Einstein's development of the relationship between "time" and simultaneous events.

> If we want to describe the motion of a particle, we give the values of its coordinates as functions of time. However, we must keep in mind that a mathematical description of this kind only has physical meaning if we are already clear as to what we understand here by "time." We have to bear in mind that all our judgements involving time are always judgements about simultaneous events. If, for example, I say that "the train arrives here at 7 o'clock," that means, more or less, "the pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events."[24]

Einstein's use of quotation marks in this excerpt is more straightforward. The word "time" is enclosed in quotation marks to call special attention to it. The phrase "the train arrives here at 7 o'clock," is a half-direct and half-indirect quotation and the quotation marks are used to emphasize the phrase. (Notice there is no comma, and the word the is not capitalized.) In this passage, the final use of quotation marks is for the definition "the pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events," which defines the passage's previous quotation.

Einstein is linking our understanding of time to our understanding of simultaneous events. In this passage the simultaneous events occur at the same location. In the next passage Einstein explains how to evaluate events that do not occur at the same location. This explanation will have a very significant bearing on the meaning of the "rest system."

> If there is a clock at point A in space, then an observer located at A can evaluate the time of events in the immediate vicinity of A by finding the positions of the hands of the clock that are simultaneous with these events. If there is another clock at point B that in all respects resembles the one at A, then the time of events in the immediate vicinity of B can be evaluated by an observer at B. But it is not possible to compare the time of an event at A with one at B without further stipulation. So far we have defined only an "A-time" and a "B-time," but not a common "time" for A and B. The latter can now be determined by establishing by definition that the "time" required for light to travel from A to B is equal to the "time" it requires to travel from B to A. For, suppose a ray of light leaves from A for B at "A-time" t_A, is reflected from B toward A at "B-time" t_B, and arrives back at A at "A-time" t'_A. The two clocks are synchronous by definition if $t_B - t_A = t'_A - t_B$.
> We assume that it is possible for this definition of synchronism to be free of contradictions, and to be so for arbitrarily many points. . . .[25]

When Einstein describes the "rest system" as a system where, "the 'time' required for light to travel from A to B is equal to the 'time' it requires to travel from B to A," he must be describing a system at absolute rest. Yet, there is a caveat that must be noted. Einstein prefaces the previous quotation with the italicized phrase, by definition. The quotation can be reformulated as, "[The common 'time' for A and B is] determined by establishing by definition that the 'time' required for light to travel from A to B is equal to the 'time' it requires to travel from B to A."

If the "rest system" was at absolute rest, the common time for points A and B would not need to be established by definition. Einstein states, "We assume that it is possible for this definition of synchronism to be free of contradictions. . ..” Does he mean free from contradictions in the “rest system” only, or does he mean free from contradictions for both a system at rest and a system in motion? Twelve paragraphs later, Einstein tells us that, for a system in motion, a light beam making a round-trip journey from point A to point B and back to point A does not adhere to his requirements for the definition of a common time for points A and B in the “rest system.” The ambiguity continues in the next passage.

> By means of certain (imagined) physical experiments, we have established what is to be understood by synchronous clocks at rest relative to each other and located at different places, and thereby obviously arrived at definitions of “synchronous” and “time.”[26]

“Clocks at rest relative to each other,” may either refer to clocks in motion or clocks at absolute rest. Passengers on a plane who are sitting in their seats are at relative rest to each other, but of course they are in motion. Clocks at absolute rest would be at rest relative to each other, as well. Einstein states, “By means of certain (imagined) physical experiments, we . . . arrived at definitions of “synchronous” and “time.” He does not restrict these definitions to the “rest system” in this passage, which is in marked contrast to the following passage.

> It is essential that we have defined time by means of clocks at rest in the rest system; because the time just defined is related to the system at rest, we call it “the time of the rest system.”[27]

In this passage, Einstein concludes that his definition of time is more appropriately understood as “the time of the rest system.” Einstein suggests in this passage that his definition of time is limited to the rest system. When Einstein states, “we call it ‘the time of the rest system,’” is he implying that a system in motion will have another definition of time? If so, that notion would contradict a prior statement he has made about the definition of the time of the rest system, “We assume that it is possible for this definition of synchronism to be free of contradictions. . ..”[28]

We will now turn our attention to the second section of “A. Kinematic Part,” which is entitled, “2. On the Relativity of Lengths and Times.” The following three passages appear in the same order of occurrence as they do in the text of Einstein’s paper. We begin with an excerpt from the opening paragraph of the second section.

> The following considerations are based on the principle of relativity and the principle of the constancy of the velocity of light.[29]

Were the notions examined in the first section also based on the principle of relativity and the principle of the constancy of the velocity of light? Why did Einstein wait until the opening paragraph of the second section to explicitly state that his two postulates formed the foundation of his argument? The constancy of the velocity of light is a necessary postulate for the formation of his definition of a common time for point A and point B, which he puts forth in the first section. Perhaps, Einstein’s two postulates were not in play in the first section. This would give us another explanation for the fact that his definition of time could only be established by definition. Einstein’s definition of a common time should be based on the principle of the constancy of the velocity of light. If the “rest system” was not at absolute rest and in fact was in motion, the principle of the constancy of the velocity of light would pose an obstacle for Einstein’s definition of a common time. Without the introduction of the principle of the constancy of the velocity of light, the “rest system” can be in motion, and this motion will be less of an obstacle to Einstein’s definition of a common time as given in the first section because a light beam with a non constant velocity is a loophole of sorts.

The next passage lays the groundwork for the thought experiment in which Einstein claims to demonstrate the relativity of simultaneity.

> Take a rigid rod at rest; let its length, measured by a measuring rod that is also at rest, be l. Now imagine the axis of the rod placed along the X-axis of the rest coordinate system, and the rod then set into uniform parallel translational motion (with velocity v) along the X-axis in the direction of increasing x.[30]

The purpose of the phrase, “along the X-axis in the direction of increasing x,” is to give a direction to the rod’s velocity. It is similar to stating the rod is traveling due east. The rigid rod’s motion is similar to a train traveling on an endless length of straight track at a constant velocity. The next passage elaborates on this rigid rod “set into uniform parallel translational motion.”

> Further, we imagine the two ends (A and B) of the rod equipped with clocks that are synchronous with the clocks of the rest system, i.e., whose readings always correspond to the “time of the system at rest” at the locations the clocks happen

> to occupy; hence, these clocks are "synchronous in the rest system."[31]

Notice that Einstein does not describe the method by which the clocks in the moving system of the rigid rod are to be synchronized with the clocks in the "rest system." He states only that the clocks in the moving system are synchronized with the clocks in the "rest system." We know that according to Einstein's definition of common time, clocks are synchronized when the time it takes a light beam to travel from point A to point B is equal to the time it takes a light beam to travel from point B to point A.

Let us suppose point A is located in the "rest system" and point B is located at one end of the rigid rod. The rigid rod is designated as the moving system by Einstein. Along with the clock that Einstein indicates is at point B, let us also suppose there is a mirror. The mirror is connected to the clock so that when a powerful light beam strikes the mirror the clock will instantly note the time of this event. At point A there is a clock and a device that can project a powerful light beam to the mirror on the moving rigid rod. The clock at point A is connected to the light-beam projecting device so that it will instantly note the time when the light beam begins its journey to the mirror at point B. The clock at point A is also connected to another device that can detect the returning light beam from the mirror at point B. The instant this device detects the light beam returning from the mirror, the clock at point A notes the time of this event, as well. According to Einstein the clocks at points A and B are synchronized if the following equation is valid: $t_B - t_A = t'_A - t_B$.

This equation is only valid if the “rest system” is at absolute rest and the light beam begins its journey from the “rest system.” It is only under these conditions that the distances the light beam travels in each leg of its journey are equal. The moving system of the rigid rod is traveling away from point A with a constant velocity. When the light beam from point A strikes the mirror at point B, it has traveled a distance d. Since the light beam is reflected instantly from the mirror, when it returns to point A, the return distance it has traveled is the same as its outbound distance, i.e., distance d.

Next, let us change the experimental set up. Now, the light beam is projected from point B of the moving system of the rigid rod and the mirror has been relocated to point A of the “rest system.” When the light beam from point B strikes the mirror relocated to point A, it has traveled a distance d. When the light beam returning from the mirror strikes the detection device relocated to point B, it has traveled the distance $d + v\,t'_B$. The distance d is the distance that separated point B from point A at the instant the light beam began its journey to point A. Since that instant, the moving system of the rigid rod has been moving further away from point A with a constant velocity v. While the light beam was traveling from point B to point A, the rigid rod was traveling further from point A. Also, while the light beam was returning from point A to point B, the rigid rod was traveling further from point A. The total increase in the distance of point B from point A is the velocity of the rigid rod times the duration of the light beam’s round-trip journey or $v\,t'_B$. We are assuming the clock at point B read 00:00 at the beginning of the light beam’s journey.

The time it takes the light beam to travel from point B to point A is the following: $t_A - t_B = d/ V$. The time it takes the light beam to return from point A to point B is the following: $t'_B - t_A = (d + v\, t'_B)/ V$. When the light beam travels from point B on the moving system to point A on the rest system and back again to point B, Einstein's definition for synchronized clocks, $t_A - t_B = t'_B - t_A$, is not met. Instead, an inequality is produced that is represented by the following: $t_A - t_B < t'_B - t_A$.

The conclusion of this analysis is that the clocks in the moving system of the rigid rod can be synchronized with the clocks in the "rest system." However, there are two important stipulations: first, the "rest system" must be at absolute rest, and, secondly the light ray must always originate from the rest system. The fact that the clocks in the moving system of the rigid rod can only be synchronized with the clocks in the "rest system" by modifying Einstein's definition of time is enough to undermine his claim of the relativity of simultaneity. As Einstein stated, "The two clocks are synchronous by definition if $t_B - t_A = t'_A - t_B$. We assume that it is possible for this definition of synchronism to be free of contradictions, and to be so for arbitrarily many points. . .. "[32] Since the clocks in the moving system of the rigid rod can only be synchronized with the clocks in the "rest system" by modifying Einstein's definition of time, therefore, if we strictly adhere to his definition, the clocks in the two systems cannot be synchronized. But, according to Einstein the clocks in the "rest system" are synchronized with the clocks in the moving system. The essence of Einstein's argument is that although the clocks in the moving system are synchronized with the clocks in the "rest system" the observers co-moving with the moving system will determine that their two clocks at either end of their rod are not synchronized.

> We further imagine that each clock has an observer co-moving with it, and that these observers apply to the two clocks the criterion for the synchronous rate of two clocks formulated in section 1. Let a ray of light start out from A at time t_A; it is reflected from B at time t_B, and arrives back at A at time t'_A. Taking into account the principle of the constancy of the velocity of light, we find that $t_B - t_A = r_{AB} /(V - v)$ and $t'_A - t_B = r_{AB} /(V + v)$, where r_{AB} denotes the length of the moving rod, measured in the rest system. Observers co-moving with the rod, would thus find that the two clocks do not run synchronously, while observers in the system at rest would declare them to be running synchronously.
> Thus we see that we cannot ascribe absolute meaning to the concept of simultaneity; instead, two events that are simultaneous when observed from some particular coordinate system can no longer be considered simultaneous when observed from a system that is moving relative to that system.[33]

What compels the observers co-moving with the rod to use the following definition of two synchronous clocks: $t_A - t_B = t'_B - t_A$, to determine if the clock at one end of their rod is synchronized with the clock at the other end of their rod? Einstein's closing comment in the first section on this definition of time is, "It is essential that we have defined time by means of clocks at rest in the rest system; because the time just defined is related to the system at rest, we call it 'the time of the rest system.'"[34] Observers co-moving with the rod would be justified in questioning whether Einstein's definition of synchronous clocks was appropriate for their situation. Einstein states, "we have defined time by means of clocks at rest in the rest system. . ..”[35] Clocks at rest in the rest system are essential to Einstein’s definition. The clock at each end of the rod is not at rest in the rest system. Instead, they are at rest in the moving system.

Einstein’s penultimate comment in the first section on his definition of time is, “By means of certain (imagined) physical experiments, we have established what is to be understood by synchronous clocks at rest relative to each other and located at different places, and thereby obviously arrived at definitions of ‘synchronous’ and ‘time.’”[36] What should the observers co-moving with the rod make of the above statement? Their rod has a clock at each end. Their clocks are at rest relative to each other, and they are located in different places. The observers co-moving with the rod might conclude that Einstein’s definition of time does apply to their situation. However, after reflection, the observers co-moving with the rod might conclude that it was ambiguous whether or not Einstein meant his definition of time to apply to their circumstance.

What conclusion would the observers co-moving with the rod arrive at if they applied Einstein’s definition of time in the following thought experiment? The observers assume that their two clocks are synchronized. Next, they follow Einstein’s criterion for determining if two clocks are synchronized. Einstein gives an encapsulated version of their procedure and results when he states, “Let a ray of light start out from A at time t_A; it is reflected from B at time t_B, and arrives back at A at time t'_A. Taking into account, the principle of the constancy of the velocity of light, we find that $t_B - t_A = r_{AB} /(V - v)$ and $t'_A - t_B = r_{AB} /(V + v)$, where r_{AB} denotes the length of the moving rod, measured in the rest system.”[37] The observers co-moving with the rod conclude from their thought experiment that Einstein's definition of time does not apply to their situation.

We should expand on Einstein's encapsulated version to explain how the observers reached their conclusion. In Einstein's thought experiment, the observers are using his definition of time to determine if their clocks are synchronized. When the result of their test determines that their clocks are not synchronized, the observers unquestioningly accept the results. In our thought experiment the observers question whether Einstein's definition of time applies to their circumstances. Our thought experiment is the same as Einstein's except that in our thought experiment the observers assume their clocks are synchronized. Since the observers are doing a thought experiment, this assumption is legitimate.

First, the observers will analyze the journey of the light beam from point A to point B. Their rod is moving forward, i.e., along the X-axis in the direction of increasing x. Point B is at the front of their rod, and point A is at the rear of their rod. When the light beam emanating from point A is projected toward the mirror at point B, point B is racing away from the oncoming light beam with the velocity v. The total distance the light beam must travel is the length of the rod, r_{AB}, plus the distance forward point B has traveled in the time it takes the light beam to travel from point A to the mirror at point B, which is $v(t_B - t_A)$. The total distance the light beam travels is $[r_{AB} + v(t_B - t_A)]$, the velocity of the light beam is V, and the time it takes the light beam to travel from point A to point B is $(t_B - t_A)$. The duration of the light beam's journey equals the distance traveled by the light beam divided by the speed of the light beam, $(t_B - t_A) = [r_{AB} + v(t_B - t_A)]/ V$. If we multiply each side of the equation by V and then subtract $v(t_B - t_A)$ from each side of the equation, we have $(t_B - t_A)(V - v) = r_{AB}$. If we divide each side of the equation by $(V - v)$, we have $(t_B - t_A) = r_{AB} /(V - v)$. The observer's thought experiment has produced the same equation for $(t_B - t_A)$ as Einstein's did.

In the second portion of our thought experiment the observers will analyze the return journey of the light beam from the mirror at point B to point A. When the light beam reflected from the mirror is returning to point A, point A is racing toward the oncoming light beam with the velocity v. The total distance the light beam travels in its return journey is the length of the rod, r_{AB}, minus the distance that point A has traveled toward the returning light beam in the time it takes the light beam to complete its return journey, $v(t'_A - t_B)$. The total distance the light beam travels is $[r_{AB} - v(t'_A - t_B)]$, the velocity of the light beam is V, and the time it takes to return from point B to point A is $(t'_A - t_B)$. The observer's now can use a similar method to determine that $(t'_A - t_B) = r_{AB} /(V + v)$. Their results will again be the same as Einstein's.

Thus, the observers co-moving with the rod feel compelled to abandon Einstein's definition and determine that their clocks are synchronized if the following conditions are met: when the light beam is traveling in the same direction as the rigid rod the time the light beam takes to travel from point A to point B is $r_{AB} /(V - v)$ and when the light beam is traveling in the direction opposite to that of the rigid rod the time the light beam takes to travel from point B to point A is $r_{AB} / (V + v)$.

Therefore, the observers co-moving with the rigid rod determine that their clocks are synchronized. Since they are not at rest in the rest system, but rather a system in motion, they conclude they are not bound by Einstein's definition for "the time of the rest system," which is defined "by means of clocks at rest in the rest system." Both of the sets of observers could determine that the clocks at either end of the moving rigid rod are synchronized. The set of observers co-moving with the rigid rod could use a test for synchronization appropriate for their circumstances, and the set of observers in the rest system could use a test for synchronization appropriate for their circumstances.

Another interpretation is possible for Einstein's definition of a common time for distant events. It dispenses with the notion that synchronized clocks agree in the time they keep. Einstein states, "The two clocks are synchronous by definition if $t_B - t_A = t'_A - t_B$." This definition allows two clocks to be considered synchronized even if they do not keep the same time. This notion is counter to the accepted definition of synchronized clocks.

An example should make this clear. We will begin with a rigid rod with a clock at either end. The rod is three light seconds in length or 558,000 miles long. The rod's velocity is .5V or 93,000 miles/second and it is moving along the X-axis in the direction of increasing x. Point B is in the forward position and point A is in the rear position. A light beam emanating from point A takes six seconds to reach point B because the effective velocity of the light beam is (V – .5V) or 93,000 miles/second. A light beam returning from point B will take two seconds to reach point A because the effective velocity of the light beam is (V +

.5V) or 279,000 miles/second. If a light beam emanates from point A when the clock at point A reads 12:00:00 and the clock at point B reads 11:59:58, the light beam will reach point B when the clock at point B reads 12:00:04. The clock at point A will read 12:00:06 because it is two seconds ahead of clock B. When the light beam returns to point A, the clock at point A reads 12:00:08. Thus, 12:00:04 – 12:00:00 = 12:00:08 – 12:00:04 or $t_B - t_A = t'_A - t_B$ and the two distant clocks are synchronized according to Einstein's definition although they do not keep the same time.

In the above example, we are given the velocity of the rigid rod, which is 93,000 miles/second along the X-axis in the direction of increasing x. If we did not know the absolute velocity of the rigid rod, we could calculate it once we determined that clock A is two seconds ahead of clock B. This would tell us that the light beam requires six seconds to travel from point A to point B. The velocity of light is constant, therefore 6 sec. · (186,000 miles/sec.) = 1,116,000 miles = 558,000 miles + v · 6 sec. If we subtract 558,000 miles from each side of the equation, it gives us 558,000 miles = v · 6sec. and dividing by 6 sec. gives us v = 93,000 miles/sec.

We should note that in the example above it takes a light beam eight seconds to complete its round-trip journey. The length of the round-trip journey is 1,116,000 miles, assuming the rod is at (absolute) rest. Since light has a constant velocity, it should take the light beam only six seconds to complete the round-trip journey. The dilation of time experienced by moving objects and the contraction of moving objects along the axis aligned with the direction of motion can eliminate this discrepancy.

Since the rod is traveling with a velocity of .5V, its clocks will run slower than our clocks on earth. When one second passes on our clocks only ¾ of a second will have passed on the rod's clocks. Therefore, when four seconds pass on our clocks only three seconds will have passed on the rod's clocks. Thus, the total time of the light beam's round-trip journey as measured by the rod's clocks will be six seconds. This means the first leg of the light beam's journey will take 4.3 seconds and the final leg will take 1.5 seconds. Under these conditions the rod's velocity will be calculated to be 63,000 miles/second. If we conclude that the rod contracts along the axis of motion by a factor of ¼ due to its velocity of .5V our calculation of the rod's velocity returns to 93,000 miles/second.

It is unclear whether or not, according to Einstein, we can ever determine that clock A is two minutes ahead of clock B. If we cannot determine the true relationship between the clocks, we cannot determine the velocity of the rigid rod. Einstein requires that the rod is, "set into uniform parallel translational motion (with velocity v) along the X-axis in the direction of increasing x."[38] This requirement seems to bar us from transporting clock B to point A and comparing it to clock A. Once we move the clock B it is no longer in uniform translational motion and it is no longer part of the moving system of the rigid rod. We cannot overcome this difficulty by placing clock A and clock B on a large turntable and spinning the turntable until the clocks exchange positions. We would then note the times of a round-trip journey of a beam of light starting at point B and compare them to our previous results. This would allow us to determine that clock B did not keep the same time as clock A, but this experiment is not allowed because the motion of the turntable would remove the clocks from the uniform translational motion of the moving system.

We will summarize Einstein's dilemma. Einstein's definition for the synchronization of distant clocks is $t_B - t_A = t'_A - t_B$. This definition is only accurate for distant clocks located on objects in a state of absolute rest, and it is incorrect for distant clocks located on moving objects, provided we employ the standard definition of synchronization. But, we should recall that Einstein claims there is no need for the concept of a state of absolute rest in his theory. When we say his definition is accurate, we mean it agrees with our notion of the synchronization of distant clocks, which is that at the same instant each clock displays the same time. This means regardless of the distance separating the clocks, if there was some being that could have instantaneous knowledge of the time displayed by each of the separated clocks, the time displayed by each clock at any particular moment would be the same.

To demonstrate the relativity of simultaneity Einstein employs a thought experiment in which the clocks on a moving rod are synchronized with the clocks in a system at rest. He states that for the two distant clocks located at either end of the moving rod (with clock B in the forward position, clock A in the rear position and the distance between the two clocks denoted as r_{AB}) the value for $t_B - t_A$ is $r_{AB}/(V - v)$ and the value for $t'_A - t_B$ is $r_{AB}/(V + v)$. Thus since $r_{AB}/(V - v)$ is larger than $r_{AB}/(V + v)$, Einstein argues that $t_B - t_A$ is larger than $t'_A - t_B$ and thus $t_B - t_A \neq t'_A - t_B$. Therefore, according to Einstein, observers co-moving with the rod conclude their clocks are not synchronized while observers in the rest system claim the clocks on the moving rod are synchronized. Thus, Einstein demonstrates the relativity of simultaneity.

Since Einstein dispenses with the concept of a state of absolute rest, we can assume the rest system is actually in motion. Let's assume the rest system was moving through space and its pilots decided they wanted to maneuver alongside the moving rod and match its velocity exactly. They wanted to come to "rest" beside the moving rod. Perhaps, the rest system should not be referred to as the "rest system" until it has come to "rest" beside the moving rod. Once the pilots accomplished this task, they decided to synchronize two of their clocks (clocks Y and Z). Clock Y happened to be in very close proximity to clock A on the moving rod, and clock Z happened to be in very close proximity to clock B on the moving rod. Once the pilots on the rest system synchronized their two clocks, they decided to synchronize clock A with clockY and clock B with clock Z since each pair of clocks was only separated by an infinitesimal distance. If the clocks on the moving rod were synchronized with the clocks of the rest system in this manner, the observers on the moving rod would conclude their two clocks were synchronized and the pilots of the rest system would agree. There would be no relativity of simultaneity.

This is so because we have demonstrated that by setting the forward clock slow by a specific amount of time we can force $t_B - t_A$ to equal $t'_A - t_B$. The specific amount of time the forward clock must clock must be set slow is determined by dividing the total duration for the round-trip journey of the beam of light by two, which equals $t'_A/2$. Then we subtract $t'_A/2$ from the duration of the first leg of the light beam's round-trip journey as calculated by using the formula $t_B - t_A = r_{AB} /((V - v)$. Thus we set clock B back by the amount of time equal to $[r_{AB} /(V - v)] - t'_A/2$.

There are many ways to question Einstein's definition of distant synchronized clocks, which is $t_B - t_A = t'_A - t_B$. We can question whether the definition is appropriate for distant clocks located on moving objects. If the definition is applied to moving objects, we can point out that clocks synchronized by this method will not agree in the time that they keep. We can also point out that the clocks in a moving system can be synchronized with the clocks in a "rest system" in a way that allows both observers in the "rest system" and observers in the moving system to agree the clocks in the moving system are synchronized according to Einstein's definition although they do not agree in the time that they keep.

In the next section we will see that Einstein changes the form of his definition of distant synchronized clocks from the form $t_B - t_A = t'_A - t_B$ to the form $½(\tau_0 + \tau_2) = \tau_1$. This form states that one-half of the duration of the light beam's round-trip journey is equal to the duration of the first leg of the light beam's journey. Einstein succinctly describes the situation, "Suppose that at time τ_0, a light ray is sent from the origin of the system k [the moving system] along the X-axis to x' and reflected from there toward the origin at time τ_1, arriving there at time τ_2; we then must have $½(\tau_0 + \tau_2) = \tau_1$. . . .."[39] It should be noted that the starting time τ_0 is added to the completion time τ_2 instead of being subtracted from it. This must mean the reading of the clock at time τ_0 is zero. With only a slight explanation Einstein further modifies this form of the equation until he arrives at an extended version of the equation, which is the following: $\frac{1}{2}\{\tau[0, 0, 0, t] + \tau[0, 0, 0, t + x'/(V - v) + x'/(V + v)]\} = \tau[x', 0, 0, t + x'/(V - v)]$. His explanation is that the extended version includes the arguments of the function τ and applies the principle of the constancy of the velocity of light in the rest system. Einstein attempts to mathematically manipulate this equation to produce the Lorentz transformation equations. He does not succeed.

Part Two

An Analysis of Section 3. Theory of Transformations of Coordinate and Time from the Rest System to a System in Uniform Translational Motion Relative to It

In the third section, Einstein's equations become quite numerous and quite complex. In order to make referring to them less cumbersome, it would be helpful if most of the equations were given a specific number. This poses a slight difficulty because the editors of Einstein's Miraculous Year did not number the equations. Since Einstein did not number the equations in his original paper, the editors were being true to the original text.

The publisher's preface to Einstein's Miraculous Year notes that the five papers produced by Einstein in 1905, "reappeared in the original German, with editorial annotations and prefatory essays, in volume 2 of the Collected Papers of Albert Einstein, an ongoing series of volumes being prepared by the Einstein Papers Project at Boston University under the sponsorship of Princeton University Press and the Hebrew University of Jerusalem. Einstein's Miraculous Year draws heavily from this volume (The Swiss Years: Writings, 1900–1909), which remains the definitive and authoritative text of all Einstein's writings of those years. . .."[40]

The English translation of that text, which appears in Einstein's Miraculous Year, is by Trevor Lipscombe, Alice Calaprice, Sam Elworthy and John Stachel. As we have noted, their translation does not attach a number to any of Einstein's equations to make references to them less cumbersome. For the numbering of Einstein's equations we have referred to Arthur I. Miller's translation that appears in the appendix of his book Albert Einstein's Special Theory of Relativity. However, it should be noted that, without providing any explanation, A. I. Miller discontinues numbering the equations in the middle of the fourth section and in the fifth section no equations are numbered. He begins numbering the equations again in Part B, but the numbering appears to be haphazard. Therefore, we have extended A. I. Miller's numbering sequence to include the final half of the fourth section and the fifth section in its entirety.

Early on in the third section we are introduced to an unusual equation that mixes coordinates designated by numbers with coordinated designated by variables. The equation $\frac{1}{2}\{\tau[0, 0, 0, t] + \tau[0, 0, 0, t + x'/V - v) + x'/V + v)]\} = \tau[x', 0, 0, t + x'/V - v)]$, which is Eq.(3.1) according to A. I. Miller's numbering scheme, is complicated. The equation is derived from a complicated thought experiment involving a moving system and a resting system. In the moving system a beam of light makes a round-trip journey. The beam of light begins its journey at a point denoted as $(\xi, \eta, \zeta) = (0, 0, 0)$ that is the origin point of a Cartesian coordinate system. The moving system is essentially a Cartesian coordinate system constructed out of metal rods. In fact, in this thought experiment, both the moving system k and the resting system K are material constructions of a Cartesian coordinate system, plus these structures are equipped with certain experimental equipment.

The light beam travels along the ξ- axis in the direction of increasing ξ until it strikes a mirror at a distance x' from the starting point. Einstein does not provide a numerical value for x'. A numerical value for x' would tell us the distance from the starting point, $(\xi, \eta, \zeta) = (0, 0, 0)$, to the mirror. In the Greek alphabet "ξ," "η " and "ζ " are lowercase letters; their English counterparts are "x," "e" and "z " respectively. The spellings for "ξ," "η " and "ζ " are xi, eta, and zeta respectively; their pronunciations are zī, āt′ə and zāt′ə respectively. There is no equivalent for the letter "y " in the Greek alphabet.

Once the light beam strikes the mirror, it is reflected back to its starting point. Einstein's Eq.(3.1) informs us that the total time of the light beam's round-trip journey, multiplied by ½ is equal to the time it takes the light beam to travel from its starting point to the mirror located at a distance x' from the origin of system k.

To calculate the total time of the round-trip journey of the light beam Einstein adds the starting time reading to the return time reading. This may seem odd. To subtract the starting time reading from the return time reading seems more natural. For instance, if I went on a walk, in which I left my home at 8:00 a.m. and returned home at 10:00 a.m., the total time of my walk is determined by subtracting the starting time reading from the return time reading. The result is that the round-trip time for my walk is two hours. If I used a stop watch to measure the duration of my walk, the starting time reading would be 00:00. In this instance, I could add the starting time reading to the return time reading to calculate the total time of my walk. I could also subtract the starting time reading from the return time reading since the starting time reading is set to zero. In determining how much time elapsed from my starting point to my turnaround point the same type of calculations would apply. Of course, if the starting time of any journey is set to zero the duration of any segment of the journey can be calculated by taking the elapsed-time reading at the moment of the completion of that particular segment. These considerations will be important when we use certain mathematical rules involving partial derivatives. Now, we will return to our description of the resting system K and the moving system k.

The place and time of any event, which is at rest in the resting system K, are described by the coordinates x, y, z, t. Time in the resting system K is denoted by t. The place and time of any event, which is at rest in the moving system k, are described by the coordinates ξ, η, ζ, τ. Time in the moving system is denoted by τ.

The moving system k is moving with a constant velocity, which we will denote with the letter v. The moving system k is traveling in the direction of increasing x of the resting system K. The "X-axis" of the moving system is called the ξ-axis, and it coincides with the X-axis of the resting system. The "Y-axis" and the "Z- axis" of the moving system are called the η-axis and the ζ-axis respectively. They are parallel to their counterparts, the Y and Z axes, of the system at rest.

Because of the way the two systems were aligned before system k began to move and because of the nature of their new alignment since system k began moving, we can say $x' = x - (v \cdot t)$. This means that a point with the value x on the X-axis of the resting system will coincide with a point with the value x' on the ξ-axis of the moving system at a given time t of the resting system. The value of x' will be smaller than the value of x by an amount equal to $v \cdot t$. Because of the nature of the alignment of the moving system k with the resting system K, a point at rest in the k system belongs to the system of values x', y, z, independent of time. Einstein defines τ (time in the moving system k) as a function of x', y, z, and t. Therefore, τ is a function with four variables, τ(x', y, z, t).

In mathematics a function of four variables would typically be written in this manner: $f(x, y, z, t) = x^2yz - 3z + t^3$ or some other combination of variables. In this instance, the right side of the equation, $x^2yz - 3z + t^3$, is the rule that associates to each foursome of values for the variables a number or numbers. Einstein does not give us the rule, i.e., the right side of the equation that associates to each foursome of values for the variables a particular number or numbers. Instead, he tells us the relationship between three sets of values for the variables x', y, z, and t. The three sets of values are the following: (0, 0, 0, t) and (0, 0, 0, t + x'/V − v) + x'/V + v)) and (x', 0, 0, t + x'/V − v)).

If Einstein set the function τ(x', y, z, t) equal to some grouping of variables, we could find the partial derivative of function τ(x', y, z, t) for each of the variables x', y, z, and t. We could then evaluate each of the four partial derivatives for each of the three sets of values for the variables x', y, z, and t. It should be noted that the three sets of values for the variables are a mixture of numbers and variables. So it could be a situation such that even if Einstein gave us an equation for the function τ(x', y, z, t) we still could not satisfactorily evaluate the partial derivatives.

According to Einstein if the term x' in Eq.(3.1) is allowed to become "infinitesimally small,"[41] the result is: $\frac{1}{2}[(1/(V - v) + 1/(V + v))\partial\tau/\partial t] = \partial t/\partial x' + [1/(V - v)]\partial\tau/\partial t$, which is Eq.(3.2). In Arthur I. Miller's book Albert Einstein's Special Theory of Relativity, he states, on page 209, "Einstein took x' to be infinitesimal and expanded both sides of Eq. (3.1) as a series in x'. Neglecting terms higher than first order, the result is Eq. (3.2)."[42] The exact mathematical steps indicated by A. I. Miller's explanation are difficult to ascertain.

However, it is possible that Einstein misapplied certain mathematical rules in his analysis of Eq.(3.1). These rules are used in calculations involving the partial derivatives of functions with two or more variables. Unfortunately, the following explanation of the contention that Einstein did misapply certain mathematical rules is extremely lengthy.

To understand how Eq.(3.1) generates Eq.(3.2) through the misapplication of mathematical rules, we need to rewrite Eq.(3.1). In Einstein's version of Eq.(3.1) there is a slight inconsistency. The left side of the equation, which denotes the duration of a round-trip journey of a beam of light, consists of a term representing the starting time, to which a term representing the return time is added. However, the right side of the equation, which denotes the duration of a light beam's one-way journey from the staring point to the mirror, consists only of the term representing the time reading when the light beam arrives at the mirror. To be consistent the right side of the equation should consist of the term representing the starting time and to this term we should add the term representing the time reading when the light beam arrived at the mirror. This inconsistency is not an error because the starting time has been set to zero. In fact, since the starting time is set to zero it does not need to appear on either side of the equation.

However, if the starting time was not set to zero, it would need to appear on both sides of the equation. The starting time would be subtracted from the return time on the left side of the equation, and the starting time would be subtracted from the time the light beam arrived at the mirror on the right side of the equation. We are going to rewrite Eq.(3.1) in this more generalized form in order to make the application of certain mathematical rules for partial derivatives more straightforward. The rewritten Eq.(3.1) is as follows: $\frac{1}{2}\{\tau[0, 0, 0, t + x'/(V - v) + x'/(V + v)] - \tau[0, 0, 0, t]\} = \tau[x', 0, 0, t + x'/(V - v)] - \tau[0, 0, 0, t]$. Rewritten in this form it is easier to apply two specific mathematical rules for partial derivatives. To make the mathematics clearer, we are going to concern ourselves with only the variables x' and t from Eq.(3.1).

The two mathematical rules come from the textbook Calculus and Its Applications the 4th edition by Larry J. Goldstein, David C. Lay and David I. Schneider. In their descriptions of the rules, they use functions of two variables, but the rules can be generalized to apply to functions of any number of variables.

The authors state the first rule we are going to use on page 351. We can call it Rule A. The authors state the portion of the rule we are concerned with as follows: "Let $f(x, y)$ be a function of two variables. Then if k is small we have $f(a, b + k) - f(a, b) \approx \partial f/\partial y(a, b) \cdot k$."[43] We will apply Rule A to the left side of the rewritten Eq.(3.1). Rule A allows us to manipulate Eq.(3.1) without knowing the right side of the equation of the function $\tau(x', y, z, t)$, which is the particular rule associated with the function $\tau(x', y, z, t)$. It allows us to use two of the three sets of values for the variables that Einstein has provided in Eq.(3.1). There is also a similarity between Rule A' s requirement that k be small and Einstein's requirement that x' must become infinitesimally small.

Rule A is also similar to the method used to determine the duration of a light beam's round-trip journey. The duration of a light beam's round-trip journey is calculated as follows: the time reading at the end of the journey minus the time reading at the beginning of the journey. The time reading at the end of the journey is similar to the term $f(a, b + k)$, and the time reading at the beginning of the journey is similar to the term $f(a, b)$. The constant a would represent the point on the ξ-axis that coincides with both the beginning and end of the light beams round-trip journey. Although the ξ-axis is an axis of the moving system k, the ξ coordinate of the point is the same for both events, and according to Einstein's designation its numerical reading on the ξ-axis is zero. The constant b would represent the starting time for the light beam's journey, and by Einstein's designation it is also zero. It is represented in Eq.(3.1) by the quantity t. The sum (b + k) would represent the starting time plus the addition of the return time, which would be represented by k. The duration of the light beam's round-trip journey, i.e., k is represented in Eq.(3.1) by the sum $[x'/(V - v) + x'/(V + v)]$.

As we have noted, in Eq.(3.1) Einstein gives us three sets of values for the variables (x', y, z, t) : $(0, 0, 0, t)$ and $(0, 0, 0, t + x'/(V - v) + x'/(V + v))$ and $(x', 0, 0, t + x'/(V - v))$. Let us apply Rule A to the left side of the rewritten Eq.(3.1). For convenience we will exclude, for the moment, the quantity ½, which is located outside these parentheses {. . .}. The quantities within these parentheses include the first and second set of values from Eq.(3.1). We must match up the quantities in Rule A to their counterparts on the left side of the rewritten Eq.(3.1) taking into account that Rule A is written for a function with two variables. In this case $f = \tau$, $f(x, y) = \tau(x', t)$, $a = 0$, $b = 0$, $k = (x'/(V - v) + x'/(V + v))$, $f(a, b + k) = \tau(0, t + x'/(V - v) + x'/(V + v))$ and $f(a, b) = \tau(0, 0)$.

Rule A states that k must be small. This is not a precise definition for the numerical value of k. In the examples provided in the textbook Calculus and Its Applications the 4^{th} edition the value of k often equals one. The authors also state on page 378, “the approximation improves as k approaches zero.”[44] In Eq.(3.1) if we let x' assume the numerical value of one unit of length (for example, one mile), then for all values of v that are at least one unit of distance per unit of time (for example, one mile per second) less than the velocity of light, the value of k will be either approximately one or less than one. For example, if we let v equal the orbital velocity of the earth, the value for k would be approximately 0.000011 seconds. However, as v exceeds 99.999% the velocity of light, k will grow progressively larger than one. Thus, when x' assumes the value of one unit of distance, k assumes the values of either approximately one or less than one for values of v less than 99.999% the velocity of light. Therefore, if x' assumes the value of one unit of distance, then under most circumstances, when Rule A is applied to Eq.(3.1), k assumes values that are typically associated with the requirement that k should be small. This may explain why Einstein allows x' to assume the value of one although his stated requirement is that x' must become infinitesimally small. This is evidence that Einstein is applying Rule A.

Considering this lengthy explanation, applying Rule A to the left side of the rewritten Eq.(3.1) gives us the following:

Formula A

$$\tau(0, t + 1/(V - v) + 1/(V + v)) - \tau(0, t) \approx \partial\tau/\partial t(0, t) \cdot (1/(V - v) + 1/(V + v))$$

The right side of Formula A is very similar to the left side of Einstein's Eq.(3.2). We can multiply the entire left side and the entire right side of the Formula A by ½ , since the ½ in Eq.(3.1) was separated from the terms to which we applied Rule A by parentheses. This gives us the following:

Formula B

$$\tfrac{1}{2}[(0,\ t + 1/(V - v) + 1/(V + v)) - \tau(0,\ t)] \approx \tfrac{1}{2}[\partial\tau/\partial t(0,\ t) \cdot (1/(V - v) + 1/(V + v))]$$

Thus, the only differences between the right side of Formula B and the left side of Einstein's Eq.(3.2) are the term (0, t) and the ≈ sign, which replaces an = sign.

The right side of Formula B, $\tfrac{1}{2}[\partial\tau/\partial t(0, t) \cdot (1/(V - v) + 1/(V + v))]$, can be expressed as follows: The partial derivative of the function τ(x', y, z,t) with respect to the variable t is evaluated using the following values for x' and t: x' = 0 and t = t. Then this result is multiplied by [1/(V − v) + 1/(V + v)] and finally the entire result is multiplied by ½.

To further familiarize ourselves with Rule A, we should review an example where we evaluate partial derivatives at certain values. Next we will review an example that applies Rule A in a situation that is more conventional than the complex behavior exhibited by a light beam, when its motion is described from both a moving and resting frame of reference.

As a more conventional example of evaluating partial derivatives, let us evaluate the partial derivatives of the following function: $\tau(x', t) = x'^2 + 3tx'^2 + t^3$.

To find the partial derivative of a function, the function must be expressed in the above manner. The partial derivative of function $\tau(x', t)$ with respect to variable t is as follows: $\partial\tau/\partial t = 3x'^2 + 3t^2$. The term x'^2 is treated as a constant, and the derivative of a constant is zero. The term $3tx'^2$ is treated as the variable t multiplied by the constant $3x'^2$, and the derivative of t is one, which is multiplied by the constant $3x'^2$. The derivative of the term t^3 is $3t^2$. If we evaluate the partial derivative $\partial\tau/\partial t$ for the coordinates (0, t), the result is $3(0^2) + 3t^2 = 3t^2$.

Let us continue in our evaluation of the partial derivatives from this example, the partial derivative of function $\tau(x', t)$ with respect to the variable x' is $\partial\tau/\partial x' = 2x' + 6tx$. If we evaluate the partial derivative $\partial\tau/\partial x'$ for the coordinates (0, t), the result is 2(0) + 6t(0) = 0.

Let us apply Rule A in a conventional example. In this example we will find the approximate value of the following: $48 \cdot (125)^{1/3} \cdot (255)^{3/4}$. The solution is to let $f(x, y) = 48x^{\prime 1/3}\,y^{3/4}$. Since $f(a, b + k) - f(a, b) \approx \partial f/\partial x(a, b) \cdot k$, we let a = 125, b = 255 and k = −1. This gives us $f(125, 256 - 1) - f(125, 256) \approx \partial f/\partial x(125, 255) \cdot (-1)$. The partial derivative of the function $f(x, y)$ with respect to the variable y is $\partial f/\partial y = 48(3/4)\,x^{1/3}\,y^{-1/4} = 36x^{1/3}y^{-1/4}$. Evaluating the partial derivative at $\partial f/\partial x(125, 256)$ yields $36 \cdot 5 \cdot 1/4$, which equals 9/5. We multiply this value by k, and the result is −9/5. Thus we have $f(125, 256 - 1) - f(125, 256) \approx -9/5$. Since we want to solve the equation for $f(125, 256 - 1)$, we have $f(125, 256 - 1) \approx f(125, 256) - 9/5$. This gives us $(48 \cdot 125^{1/3} \cdot 255^{3/4}) \approx (48 \cdot 5 \cdot 64) - 9/5 = 15{,}358.2$. Therefore, $48 \cdot (255)^{3/4} \cdot (125)^{1/3} \approx 15{,}358.2$. Using a calculator to determine $48 \cdot (255)^{3/4} \cdot (125)^{1/3}$ yields 15,314.97799.

The above calculation are merely an example of the way Rule A can be applied to a function with two variables. The important point is that when Rule A is applied to the left side of rewritten Eq.(3.1), Rule A generates a quantity that is strikingly similar to the left side of Eq.(3.2).

Now, we will apply Rule B to the right side of the rewritten Eq.(3.1). The rewritten Eq.(3.1) is as follows: ½{ τ[0, 0, 0, t + x'/(V − v) + x'/(V + v)] − τ[0, 0, 0, t]} = τ[x', 0, 0, t + x'/(V − v)] − τ[0, 0, 0, t].

Rule B is given on page 378 of Calculus and Its Applications the 4th edition, "Therefore, it is not surprising that when both coordinates are changed, the change in f(x, y) is approximated by the sum of these terms. Precisely, we have as follows: f(a + h, b + k) − f(a, b) ≈ [∂f/∂x (a, b)] · h + [∂f/∂y (a, b)] · k, where the approximation improves as h, k approach zero. The expression on the right side is usually called a total differential. Its value depends on h and k, as well as the partial derivatives at x = a, y = b."[45]

We must match the terms of Rule B to the appropriate terms from the right side of the rewritten Eq.(3.1). The term f(x, y) = τ(x', t), the term f(a, b) = τ(0, t), and the term f(a + h, b + k) = τ(0 + x', t + x'/(V − v)). From this we can determine that a = 0, b = t, h = x', and k = x'/(V − v). The term ∂f/∂x = ∂τ/∂x' and the term ∂f/∂y = ∂τ/∂t. When we match up the terms from Rule B to the right side of the rewritten Eq.(3.1), there is an insignificant discrepancy. The term (a + h) is matched to the term (0 + x') while in the rewritten Eq.(3.1) it appears as the term (x').

Applying Rule B to the right side of Eq.(3.1) gives us the following:

Formula C

$$\tau(0 + x', t + x'/(V - v)) - \tau(0, t) \approx [\partial\tau/\partial x' (0, t)] \cdot x' + [\partial\tau/\partial t (0, t)] \cdot x'/(V - v).$$

If we follow the convention we established previously that x' should be set equal to one so that both h and k are small, we can transform the right side of Formula C so that it is very similar to the right side of Eq.(3.2). As we stated previously Einstein required that x' become infinitesimally small, yet on both the left and right sides of his Eq.(3.2) the term x' was changed into one. This is evidence that Einstein is applying Rules A and B. If we allow x' to become one, the formula becomes the following:

Formula D

$$\tau(0 + 1, t + 1/(V - v)) - \tau(0, t) \approx [\partial\tau/\partial x' (0, t)] \cdot 1 + [\partial\tau/\partial t (0, t)] \cdot 1/(V - v).$$

The right side of the formula can be stated as follows: The partial derivative of the function $\tau(x', t)$ with respect to the variable x' is evaluated for the coordinates (0, t). This result of the evaluation is multiplied by one. To this result, we add another quantity. That quantity is the following: the partial derivative of the function $\tau(x', t)$ with respect to the variable t, evaluated for the coordinates (0, t), and with the result of this evaluation multiplied by $1/(V - v)$.

Perhaps, using another pair of parentheses would make the right side of the formula clearer. Thus, it would appear as follows: $\{[\partial\tau/\partial x' (0, t)] \cdot 1\} + \{[\partial\tau/\partial t (0, t)] \cdot 1/(V - v)\}$. We should note that multiplying the term $[\partial\tau/\partial x' (0, t)]$ by one, leaves the term unchanged, giving us $[\partial\tau/\partial x' (0, t)]$. Thus, generating the following formula:

Formula E

$$\tau(0 + 1, t + 1/(V - v)) - \tau(0, t) \approx \partial\tau/\partial x' (0, t) + \partial\tau/\partial t (0, t) \cdot [1/(V - v)].$$

Except for the appearance of the coordinates (0, t) and the ≈ sign, the right side of Formula E is the same as the right side of Einstein's Eq.(3.2) which is as follows: $= \partial\tau/\partial x' + [1/(V - v)]\, \partial\tau/\partial t$. The similarity between the right side of Eq.(3.2) and the right side of Formula E, which is produced by applying Rule B to the right side of Eq.(3.1), is striking.

There are three remaining areas that need to be expanded upon to give a complete picture of the application of Rules A and B to Eq.(3.1). First, Rules A and B must be expanded from rules that apply to functions with two variables to rules that apply to functions with four variables. Secondly, since the coordinates (0, t), which appear once when Rule A is applied and twice when Rule B is applied, are the only terms that separate the above results from Einstein's results, we must determine whether these terms can be deleted from right sides of Formulas B and E so that they more closely replicate Einstein's results. Thirdly, the significance of the use of the ≈ sign as opposed to an = sign must be explored.

First, the Rules A and B can be expanded to accommodate functions with four variables. The expanded Rule A is as follows: Let $f(x, y, z, t)$ be a function of four variables. If k is small, we have $f(a, b, c, d + k) - f(a, b, c, d) \approx \partial f/\partial t\, (a, b, c, d) \cdot k$. The expanded Rule B is as follows: Let $f(x, y, z, t)$ be a function of four variables. If both h and k are small, we have $f(a + h, b, c, d + k) - f(a, b, c, d) \approx [\partial f/\partial x(a, b, c, d)] \cdot h + [\partial f/\partial t(a, b, c, d)] \cdot k$. The result of expanding Rules A and B is that the term (0, t) is enlarged from two coordinates to four coordinates, and it becomes (0, 0, 0, t).

Secondly, the term (0, t) and the term in its expanded form (0, 0, 0, t) cannot be deleted from Formulas B and E. If Einstein applied Rules A and B to his Eq.(3.1) to produce Eq.(3.2), he deleted the term (0, 0, 0, t) to suit his own purposes.

Thirdly, the introduction of the ≈ sign undercuts Einstein's argument that the left side of Eq.(3.2) is equal to the right side of Eq.(3.2). By applying Rule A, we produce a quantity that is approximately equal to the left side of Eq.(3.1). By applying Rule B, we produce a quantity that is approximately equal to the right side of Eq.(3.1). These two approximations are not equal to each other. It is true that the approximation of the left side of Eq.(3.1) improves as k approaches zero. In this instance k is equal to $(x'/(V + v) + x'/(V - v))$. It is also true that the approximation of the right side of Eq.(3.1) improves as both h and k approach zero. In this case h is equal to x' and k is equal to $x'/(V - v)$. It may be confusing that the term k appears both in Rule A and Rule B. It does not mean that the term k must assume the same value in both Rule A and Rule B. The use of the term k in both rules was merely a matter of convenience for the authors of Calculus and Its Applications 4^{th} edition.

The term k employed for Rule A, which was applied to the left side of Eq.(3.1), produces a much more accurate approximation than the term h employed for Rule B, which was applied to the right side of Eq.(3.1). Therefore, the approximation of the left side of Eq.(3.1) is much more accurate than the approximation of the right side of Eq.(3.1). Thus, the left side approximation of Eq.(3.1) is not equal to the right side approximation of Eq.(3.1). The formula these approximations generate, Eq.(3.2), is not an equality. The left side of the formula is only approximately equal to the right side of the formula. Einstein's requirement that x' must become infinitesimally small does not alter the fact that Eq.(3.2) is not an equality. No matter how infinitesimally small x' becomes the value of the term k, which is generated for the left side of the Eq.(3.1), will be much closer to zero than the term h, which is generated for the right side of Eq.(3.1). Therefore, the approximation produced of the left side of Eq.(3.1) will be more accurate than the approximation produced of the right side of Eq.(3.1) because the right side approximation employs the term h, which is not as close to zero as the term k. This remains so even if the term h is infinitesimally small.

The following example will illustrate the points made in the previous argument. Let v represent the orbital velocity of the earth, which is about 19 miles/second. The term V represents the velocity of light, which is about 186,000 miles/second. The term x' represents the distance the beam of light in Einstein's thought experiment travels from the starting point until it strikes the mirror. For convenience, let us set x' equal to one mile.

We applied Rule A to the left side of Eq.(3.1); the term k for Rule A is as follows: $k = x'/(V + v) + x'(V - v) = 1_{mile}/(186{,}000_{miles/sec.} + 19_{miles/sec.}) + 1_{mile}/(186{,}000_{miles/sec.} - 19_{miles/sec.}) = 0.000011$ seconds.

We applied Rule B to the right side of Eq.(3.1); the term h for Rule B was as follows: h = x' = 1 mile.

The value for the term k for Rule A is about 90,000 times smaller than the value for the term h for Rule B. Since the value for the term k is smaller than the value for the term h, the approximation of the left side of Eq.(3.1) is more accurate than the approximation of the right side. This is an instance where you can mix apples and oranges or, at least, seconds and miles. However, the role that term k for Rule B plays in these calculations should be examined.

As we noted before, we applied Rule B to the right side of Eq.(3.1), using the term k for Rule B. The term k for Rule B is as follows: k = x'/(V − v) = $1_{mile}/(186{,}000_{miles/sec.} - 19_{miles/sec.})$ = 0.0000054 seconds.

The value for the term k for Rule B is about twice as small as the value of the term k for Rule A. However, this is overshadowed by the fact that the value of the term k for Rule A is about 90,000 times smaller than the value of the term h for Rule B. Nothing in this example suggests that there is an equality between the value of the term k for Rule A and the values of the terms h and k for Rule B. Such an equality would be needed to suggest that the approximation of the left side of Eq.(3.1) is equal to the approximation of the right side of Eq.(3.1).

The value of the term k for Rule A will always be about 90,000 times smaller than the value of term h for Rule B. This is the case even if the term h for Rule B, i.e., x', becomes infinitesimally small. This shows that Eq.(3.2), $\frac{1}{2}[1/(V - v) + 1/(V + v)]\partial\tau/\partial t = \partial\tau/\partial x' + [1/(V - v)]\,\partial\tau/\partial t$, is not an equality; the equation is only an approximate equality. The equation should be expressed as follows: $\frac{1}{2}[1/(V-v) + 1/(V+v)]\partial\tau/\partial t \approx \partial\tau/\partial x' + [1/(V-v)]\,\partial\tau/\partial t$. The right side of the equation is about 90,000 times more accurate an approximation than the left side of the equation. Further, the approximate equality is only valid if we accept that Eq.(3.1) is valid. Also, the approximate equality is only valid for the value of x' specified in the approximate equality. The specified value of x' in this instance is $x' = 1$ unit of distance.

The conclusion of this entire analysis is that Einstein's transformation of Eq. (3.1) into Eq. (3.2) is flawed. The mathematics, he most likely employs, do not generate formulas that are equalities, instead they are designed to generate formulas that are approximate equalities. His requirement that x' be chosen infinitesimally small is remarkable because that requirement is clearly not satisfied.

His argument is analogous to the kind of mistake a bicyclist could make. Let us imagine that a bicyclist is going on a ten-mile round-trip journey. During the outbound half of his journey, he is heading west and riding into a steady, 20 mph wind from out of the west. During the return half of his journey, he is traveling east, and riding now with the same, steady wind at his back.

To test his bicycle's brakes and tire inflation, before he begins his journey, he travels back and forth over a 30-yard distance several times. He travels west 30 yards, then he returns traveling east 30 yards. He concludes that his bicycle is in good condition, and he also concludes that the wind will not affect his journey. However, a few miles into his journey, he realizes his second conclusion is wrong. The wind will affect his journey. When he was traveling back and forth over a 30-yard distance, the slight distance he traveled gave him the wrong impression that the wind would not affect his journey. He concludes that when he was traveling back and forth over the 30-yard distance the wind did affect him, but its affect was too small for him to notice.

This is the kind of argument Einstein is making; however, Einstein never acknowledges that his conclusion is incorrect. He maintains that if the round-trip distance, which a light beam travels, is infinitesimally small, then the motion of the of the body, on which the light beam experiment is taking place, will have no effect on the duration of the outbound segment or the duration of the return segment of the light beam's journey.

We can make an argument that supports Einstein's conclusion. If we say that infinitesimally small refers to a distance that is zero units in length or effectively zero units in length for all possible calculations, then Einstein's argument is correct, but all this means is the following: on a moving body, when a beam of light makes a round-trip journey of zero units in length the duration of each leg of the journey is equal. The statement may be true, but more importantly it is pointless because the statement claims that the light beam has made a round-trip journey when in fact it has made no journey, whatsoever. This kind of argument should not serve as a template for analyzing the behavior a light beam making a round-trip journey on a moving body.

Is there another way to derive Eq.(3.2) from Eq.(3.1)? If a way could be found to derive Eq.(3.2) so that its equal sign was legitimate, it would strengthen Einstein's argument. A significant barrier to overcome is the fact that Eq.(3.1) is not in form that can undergo partial differentiation, yet Eq.(3.2) contains two partial derivatives, $\partial\tau/\partial x'$ and $\partial\tau/\partial t$. Eq.(3.1) consists of three quartets (groups of four) of coordinates for the function $\tau(x', y, z, t)$. The function $\tau(x', y, z, t)$, which is composed of the four variables x', y, z, and t, needs a rule (an equation) that associates to each quartet of values for the variables a number in order for the function to be partially differentiated. Einstein does not supply us with such a rule.

As we mentioned before, on page 209 of Arthur I. Miller's Albert Einstein's Special Theory of Relativity he provides an explanation of the derivation of Eq.(3.2) from Eq.(3.1). He states, "Toward obtaining differential equations whose solutions provide the functional dependence of τ on (x', y, z, t), Einstein took x' to be infinitesimal and expanded both sides of Eq.(3.1) as a series in x'. Neglecting terms higher than first order the result is $\partial\tau/\partial x' + v/(V^2 + v^2)\,\partial\tau/\partial t = 0$." [46]

First, we should explain why the equation $\partial\tau/\partial x' + v/(V^2 + v^2)\,\partial\tau/\partial t = 0$ does not look like the Eq.(3.2) we are familiar with. The equation $\partial\tau/\partial x' + v/(V^2 + v^2)\,\partial\tau/\partial t = 0$ is Einstein's Eq.(3.3), which is derived from Eq.(3.2) by finding a common denominator for the fractions and grouping the partial derivatives on the left side of the equation. The procedure is as follows: Eq.(3.2), $\frac{1}{2}[1/(V - v) + 1/(V + v)]\partial\tau/\partial t = \partial\tau/\partial x' + [1/(V - v)]\,\partial\tau/\partial t$, with a common denominator for the fractions becomes, $\frac{1}{2}[2V/(V^2 - v^2)]\partial\tau/\partial t = \partial\tau/\partial x' + [(V + v)/(V^2 - v^2)]\,\partial\tau/\partial t$. The previous equation with the term $[V/(V^2 - v^2)]\partial\tau/\partial t$ subtracted from both sides is, $0 = \partial\tau/\partial x' + [(V + v)/(V^2 - v^2)]\,\partial\tau/\partial t - V/(V^2 - v^2)]\partial\tau/\partial t$ (Note: $\frac{1}{2} \cdot 2 = 1$). The previous equation with the terms that contain $\partial\tau/\partial t$ grouped together is, $0 = \partial\tau/\partial x' + [\, v/(V^2 - v^2)]\, \partial\tau/\partial t$, and finally expressing the equation with the terms containing partial derivatives placed on the right side of the equation gives us Eq.(3.3).

With that explanation concluded, we can explore the notion of expanding both sides of Eq.(3.1) as a series in x'. Is it mathematically legitimate to expand both sides of Eq.(3.1) as any series in x' or are we limited to a certain group of series in x' ? Is there only one series in x' in which we can legitimately expand both sides of Eq.(3.1)?

We could expand both sides of Eq.(3.1) in many different series in x', and this would produce many different outcomes. We assume A. I. Miller requires that we only expand both sides of Eq.(3.1) in those series in x' that will produce the desired outcome. This is not a mathematically legitimate procedure, unless he acknowledges that the outcome he generates is only one among a great many possible outcomes. Or, he could be referring to a specialized series such as theTaylor's series.

For simplicity, if we ignore all the terms in Eq.(3.1) that do not directly contain the value x', we could expand the left side of Eq.(3.1) as a series in x' as follows: $[x'/(V + v) + x'/(V - v)] = 2Vx'/(V^2 - v^2)$, which could produce the following series, $2Vx'/(V^2 - v^2)$, $4Vx'/(V^2 - v^2)$, $6Vx'/(V^2 - v^2)$, . . . $2nVx'/(V^2 - v^2)$, where n = 1, 2, 3, ...

Again for simplicity, if we ignore all the terms in Eq.(3.1) that do not directly contain the value x', we could expand the right side of Eq.(3,1) as a series in x' as follows: x' could produce the following series, 1x', 2x', 3x', . . . nx', where n = 1, 2, 3, . . ., and [x'/(V – v)] could produce the following series, 1x'/(V – v), 2x'/(V – v), 3x'/(V – v), . . . nx'/(V – v), where n = 1, 2, 3, ...

A. I. Miller's next requirement is that we neglect terms higher than the first order. According to the Mathematics Dictionary edited by Glenn James and Robert C. James the word *order* has as many as twenty different mathematical definitions. Of these definitions, two seem to be applicable to Miller's scenario. The *Mathematics Dictionary* defines the "order of an algebraic curve or surface as the degree of its equation. . .."[47] While this definition may be applicable, we will ignore it in favor of the definition for differences of the first order because that definition involves sequences. We will implicitly return to the concept of the degree of an equation when we discuss a Taylor's series. The Mathematics Dictionary defines the "differences of the first order or first-order differences as the sequence formed by subtracting each term of a sequence from the next succeeding term. The first-order differences of the sequence (1, 3, 5, 7, . . .) would be (2, 2, 2, . . .)."[48] We begin by subtracting the first term of a series from the second term of a series. When we apply that procedure to the series we generated for the left side of Eq.(3.1), the result is as follows: $4Vx'/(V^2 - v^2) - 2Vx'/ (V^2 - v^2) = 2Vx'/ (V^2 - v^2)$. When we apply the same procedure to the two series we generated for the right side of Eq.(3.1), the result is as follows: 2x' – 1x' = 1x' and 2x'/(V – v) – 1x'/(V – v) = 1x'/(V – v).

This simplified version of Eq.(3.1) is as follows: $2Vx'/(V^2 - v^2) = 1x' + 1x'/(V - v)$. The entire left side of this simplified version of Eq.(3.1) is multiplied by ½ because of the ½ outside of all the parentheses of the original Eq.(3.1). This gives us the following: $Vx'/(V^2 - v^2) = 1x' + 1x'/(V - v)$. Subtracting $Vx'/(V^2 - v^2)$ from each side and finding a common denominator for the fractions gives us the following: $0 = 1x' + vx'/(V^2 - v^2)$. If we assume that the term $vx'/(V^2 - v^2)$ is a term denoting time, which it is, and therefore arbitrarily replace the term x' with the term t, the result is the following: $0 = 1x' + vt/(V^2 - v^2)$. If we further assume $\tau(x', t) = 1x' + vt/(V^2 - v^2)$, we can find the partial derivatives $\partial\tau/\partial x'$ and $\partial\tau/\partial t$. They are the following: $\partial\tau/\partial x' = 1$ and $\partial\tau/\partial t = v/(V^2 - v^2)$. This outcome is not the result we desired, but the process does highlight some of the difficulties in following A. I. Miller's strategy.

If we expand both sides of Eq.(3.1) as a series in x', how do we determine the precise form of that series? Is the Eq.(3.1) expanded into two series or three series? The series should include the term t with a coefficient that is similar to $v/(V^2 - v^2)$, but this does not appear to be possible. Somehow the series should be set equal to the function $\tau(x', t)$, but this does not appear to be possible. Somehow the terms $\partial\tau/\partial x'$ and $\partial\tau/\partial t$ should appear in the same equation, but this does not appear to be possible.

A. I. Miller may be referring to a Taylor's series when he uses the phrase expanded both sides of Eq. (3.1) as a series in x' . A Taylor's series is a complicated mathematical device that could produce an equation that does bear some resemblance to Eq. (3.2) from Eq. (3.1). Without going into a great deal of complicated mathematics we can demonstrate that Einstein is not using a Taylor's series or at least not correctly using a Taylor's series. First, we need to define Taylor’s theorem in order to understand a Taylor’s series. According to the Mathematics Dictionary Taylor’s theorem is, “A theorem which defines a polynomial whose graph runs very close to that of a given function throughout a certain interval, and a remainder which supplies a numerical limit to the difference between the ordinates of the two curves; the approximate representation of a given function on a certain interval (in the neighborhood of a certain point) by a polynomial.”[49] Taylor’s theorem is the approximate representation of a given function. Taylor’s theorem and series can be extended to functions of any number of variables. Taylor’s theorem for a function of two variables does bear some resemblance to the results produced by Einstein.

The definition of Taylor’s series dispels any notion that Einstein could use it to obtain his results. The key is that only the sum of a Taylor’s series can represent a given function, not the first member of the series. The Mathematics Dictionary states, “If n be allowed to increase without limit in the polynomial obtained by Taylor’s theorem, the result is called a Taylor’s series. The sum of such a series represents the expanded function if, and only if, the limit of R_n as n becomes infinite is zero.”[50] The important point of the quotation is the following: The sum of such a series represents the expanded function.

We now have completed our analysis of Eqs. (3.1) and (3.2), except for the fact that Einstein utilizes two other thought experiments that are nearly identical to the one we have just analyzed. He intoduces the two other thought experiments directly following Eq. (3.2). To reiterate, both of these thought experiments are similar to the one already described. They both take place on the moving system k, and the moving system k is moving along the X-axis in the direction of increasing x of the rest system K . Both thought experiments involve a beam of light traveling from a starting point to a mirror and back to the starting point. In one thought experiment the light beam travels along the η-axis, and in the other thought experiment it travels along the ζ-axis. When the light beam travels along either the η-axis or the ζ-axis, it is observed from the rest system K. The velocity of the light beam when observed from the rest system is $(V^2 - v^2)^{1/2}$. The reason for this is that although the light beam is traveling along either the η-axis or the ζ-axis it does not appear this way to the observers in the rest system. The observers in the rest system observes the light beam originating from a starting point on the X-axis and traveling diagonally to a point on either the η-axis or ζ-axis. Since the X-axis of the rest system K and the ξ-axis of the moving system k coincide, the starting point is both x and x' = x – vt. The velocity of the light beam that appears to travel along the diagonal line is V, when observed from the rest system. The diagonal line forms the hypotenuse of a right triangle. The segment of either the η-axis or the ζ-axis from the starting point in the moving system k to the location of the mirror forms one leg of the right triangle. The other leg of the right triangle is the segment of the X-axis from the starting point in the resting system K to the point on the X-axis where either the η-axis or the ζ-axis intersect the X-axis at the moment the light beam strikes the mirror. Thus, using the Pythagorean theorem, Einstein concludes, “that light always propagates along these axes with the velocity $(V^2 - v^2)^{1/2}$ when observed from the rest system.”[51] The reasoning seems to be forced. An observer on the rest system K observes, for instance, a light beam traveling “up” the η-axis of the moving system k. The observer mistakenly assumes the light beam is traveling along a line directed diagonally upward from the rest system with the velocity V. The observer calculates that the vertical or “up” component of the light beam’s velocity is $(V^2 - v^2)^{1/2}$. The observer makes this mistaken assumption for every light beam he observes traveling “up” the η-axis. Therefore, a light beam travels “up” the η-axis with the velocity $(V^2 - v^2)^{1/2}$, at least according to our mistaken observer.

Einstein applies the same kind of reasoning that he applied to a beam of light making a round-trip journey along the ξ-axis of the moving system k, to a beam of light making a round-trip journey either along the η-axis or ζ-axis of the moving system k. He states, “Analogous reasoning –applied to the η and ζ axes–yields, remembering that light always propagates along these axes with the velocity $(V^2 - v^2)^{1/2}$ when observed from the rest system, $\partial\tau/\partial y = 0$ and $\partial\tau/\partial z = 0$.” [52]

Einstein goes into no greater detail to explain the derivation of the partial derivatives listed above. Arthur I. Miller’s Albert Einstein’s Special Theory of Relativity does provide a more detailed explanation of their derivation. He rewrites Eq. (3.1) so that it represents a light beam making a round-trip journey along the η-axis of the moving system k. The round-trip journey of the light beam is viewed from the rest system K. The light beam begins its journey when system k and system K are superposed on each other, and the light beam begins its journey from the coordinate (0, 0, 0, t). The light beam travels along the η-axis until it strikes a mirror at a distance of y' from the starting point. Since the Y-axis of K is parallel to the η-axis of k, the distance y' = y. When the light beam is observed from the K system its velocity is $(V^2 - v^2)^{½}$ so the duration of the first leg of its journey is calculated by dividing the distance the light beam travels by the apparent speed of the light beam or $y/(V^2 - v^2)^{½}$. Likewise, the total duration of the light beam's journey is $2y/(V^2 - v^2)^{½}$. When observed from the rest system K the light beam not only travels along the η-axis of the k system, it also has a component of motion along the X-axis of the K system because system k is in motion along the X-axis. For the first leg of the light beam's journey, the distance it travels along the x-axis is the velocity of system k multiplied by the duration of the first leg of the journey or $v[y/(V^2 - v^2)^{½}]$. Likewise, for the light beam's round-trip journey, the distance it travels along the X-axis is $v[2y/(V^2 - v^2)^{½}]$.

Therefore, Eq.(3.1) can be reformulated to describe the round-trip journey of a light beam along the η-axis of the moving system k as follows: $\tau[\, vy/(V^2 - v^2)^{½}, y, 0, t + y/(V^2 - v^2)^{½}] = ½[\, \tau(0, 0, 0, t) + \tau(2\, vy/(V^2 - v^2)^{½}, y, 0, t + 2y/(V^2 - v^2)^{½}]$. Arthur I. Miller states, "The result of an expansion of Eq.(6.17) [the equation above] for y considered as infinitesimal is $\partial\tau/\partial y = 0$." [53] For the reasons given before, this does not seem likely.

If we apply Rule B to the left side of the equation and Rule A to the right side of the equation, we can obtain results very similar to Einstein's results.

We must rewrite the left side of the equation as follows: $\tau(\, vy/(V^2 - v^2)^{½}, y, 0, t + y/(V^2 - v^2)^{½} - \tau(0, 0, 0, t)$. We ignore the x and z (noting that the x coordinate is $vy/(V^2 - v^2)^{½}$). We rewrite the y coordinate of the first term as 0 + y and apply Rule B, which gives us the following: $\tau(0 + y, t + y/(V^2 - v^2)^{½}) - \tau(0, t) \approx \partial\tau/\partial y(0, t) \cdot y + \partial\tau/\partial t\,(0,t) \cdot y/(V^2 - v^2)^{½}$.

We must rewrite the right side of the formula above as follows: $½[\tau(2\,vy/(V^2 - v^2)^{½}, 0, 0, t + 2y/(V^2 - v^2)^{½}) - \tau(0, 0, 0, t)]$. We ignore the x and z coordinates (noting the x coordinate is $2vy/(V^2 - v^2)^{½}$). We apply Rule A, which gives us the following: $½[\tau(0, t + 2\,y/(V^2 - v^2)^{½} - \tau(0, t)] \approx ½\,[\partial\tau/\partial t(0, t) \cdot 2y/(V^2 - v^2)^{½}]$.

Next we recombine the approximation of the left side of the equation and the approximation of the right side of the equation, which gives us the following: $\partial\tau/\partial y(0, t) \cdot y + \partial\tau/\partial t\,(0,t) \cdot y/(V^2 - v^2)^{½} \approx \partial\tau/\partial t\,(0,t) \cdot y/(V^2 - v^2)^{½}$. We should note that the approximation on the right side of the formula was multiplied by ½, and thus, the 2 (from the quantity $2y/(V^2 - v^2)^{½}$) and the ½ itself disappear from the right side of the formula. We now subtract $\partial\tau/\partial t\,(0,t) \cdot y/(V^2 - v^2)^{½}$ from each side, which gives us the following: $\partial\tau/\partial y(0, t) \cdot y \approx 0$. We assume that y does not equal zero because it represents the distance the light beam travels on the first leg of its journey, and therefore $\partial\tau/\partial y(0, t) \approx 0$. As before, our result is similar to Einstein's result of $\partial\tau/\partial y = 0$.

The same considerations can be applied to a light beam traveling along the ζ-axis of the moving system k, and the result is $\partial\tau/\partial z(0, t) \approx 0$, which again is similar to Einstein's result of $\partial\tau/\partial z = 0$.

The conclusion of the entire preceding analysis– very lengthy though parts of it were–can be stated quite succinctly. The three partial derivative equations, $\partial\tau/\partial x' + v/(V^2 + v^2)\partial\tau/\partial t = 0$, $\partial\tau/\partial y = 0$, and $\partial\tau/\partial z = 0$, which Einstein derived from Eqs. (3.1) and (3.2) in the case of the first equation and from equations analogous to Eqs. (3.1) and (3.2) in the case of the second and third equations, are invalid. We can now turn our attention to Einstein's further interpretation of these three equations.

According to Einstein, the following three equations $\partial\tau/\partial x' + v/(V^2 + v^2)\partial\tau/\partial t = 0$, $\partial\tau/\partial y = 0$, and $\partial\tau/\partial z = 0$, are the partial derivatives of a specific equation. Einstein claims the above equations "yield, since τ is a linear function, $\tau = a[t - v/(V^2 + v^2) \cdot x']$, where a is a function $\varphi(v)$ as yet unknown, and where we assume for brevity that at the origin of k we have $t = 0$ when $\tau = 0$."[54] It is not clear that an equation with four variables can be considered a linear equation. In fact it is difficult to conceive that an equation with as few as two variables, $f(x, y)$, could be a linear equation. It is similar to claiming that an equation with one variable could refer to a single point. For example, the linear equation $y = 3$ has a y value of 3 for every value of x found on the X-axis. For every value of x on the X-axis, we must rise perpendicularly three units to find the value of y, and thus, a line is generated. The authors of Calculus and Its Applications state, "A function $f(x, y)$ of two variables may be graphed in a manner analogous to that for functions of one variable. It is necessary to use a three-dimensional coordinate system, where each point is identified by three coordinates (x, y, z). For each choice of x, y, the graph of $f(x, y)$ includes the point $(x, y, f(x, y))$. The graph of $f(x, y)$ is thus a surface in three-dimensional space."[55] Thus for each choice of the coordinates (t, x') the graph of $\tau(t, x')$ would include the point $(t, x', \tau(t, x'))$ and so it would be a surface in three-dimensional space and not a line.

The equation $\tau = a[t - v/(V^2 + v^2) \cdot x']$ is denoted as Eq. (3.4), and if we calculate its partial derivatives, they do not match the three partial derivative equations that, according to Einstein, are integrated in some fashion to yield Eq. (3.4). The partial derivatives of Eq. (3.4) are $\partial\tau/\partial t = a$ and $\partial\tau/\partial x' = a[v/(V^2 + v^2)]$. Since Eq. (3.4) does not contain the variables y and z, it is legitimate to conclude $\partial\tau/\partial y = 0$, and $\partial\tau/\partial z = 0$.

However, calculating the partial derivatives of the following ungainly and highly questionable equation, $\tau = [-v/(V^2 + v^2)\partial\tau/\partial t \cdot x'] - [v/(V^2 + v^2)\,\partial\tau/\partial x' \cdot t]$ would yield the three equations, which when integrated in some fashion, according to Einstein, produced Eq. (3.4). The two partial derivatives on the right side of this equation should be treated as constants when calculating the partial derivatives of the equation. Also, for convenience the first of Einstein's three partial derivative equations, $\partial\tau/\partial x' + v/(V^2 + v^2)\partial\tau/\partial t = 0$, should be rewritten as two separate equations, $\partial\tau/\partial x' = -v/(V^2 + v^2)\partial\tau/\partial t$ and $\partial\tau/\partial t = -v/(V^2 + v^2)\partial\tau/\partial x'$. Although Einstein has not given us a specific value for the function $a = \varphi(v)$, at this point in his paper, it is unlikely the partial derivatives of Eq. (3.4) can be made to match the partial derivative equations from which Einstein claims it was generated by some form of integration.

The fact that two partial derivatives occur in the same equation i.e., $\partial\tau/\partial x' + v/(V^2 + v^2)\partial\tau/\partial t = 0$, is very strong evidence that Einstein is using Rules A and B. Normally, when calculating the partial derivatives of an equation with several variables there is a separate partial derivative equation for each variable. The form it takes will be the following: a single partial derivative sign occurs on the left side of the equation, such as $\partial\tau/\partial x'$, and the right side of the equation consists of the derivative of the variable denoted on the left side of the equation, in our example it would be the derivative of the variable x' in whatever form it occurred such as $4y^2x'^3$ while all the other variables in the equation would be treated as constants.

Now we will turn our attention to four equations that are called the transformation equations. According to Einstein, all four of these equations can be produced using Eq. (3.4), in combination with several simple equations that describe the distance a beam of light travels when it travels along each of the three axes of the moving system k. Also, we should keep in mind that Eq. (3.4) allows us to transform the time values, τ, of the moving system k into the time values, t, of the rest system K. It should be noted that from the outset we are dealing with a special case. We are not dealing with an object that could be moving with any given velocity, which would be represented by v, instead we are concerned with a light beam with the velocity V. There are other special conditions as well.

Einstein claims that Eq. (3.4), $\tau = a[t - v/(V^2 - v^2) \cdot x']$, can be transformed into Eq. (3.11), $\tau = \varphi(v)\beta(t - vx/V^2)$, where $\beta = 1/(1 - v^2/V^2)^{1/2}$ and $x' = x - vt$ and $a = \varphi(v)$; however, this claim is false. Eq. (3.11) is the first of the four transformation equations. Eq. (3.11) transforms the time values of the rest system K, which are represented by t into the time values of the moving system k, which are represented by τ. The remaining three transformation equations transform the spacial coordinates x, y, and z of the rest system K into their respective counterparts in the moving system k, which are ξ, η, and ζ.

The error in the transformation of Eq. (3.4) into Eq. (3.11) occurs when the square root is taken of only one side of the equation and not the other. For clarity we should perform the entire calculation. It will be helpful to note that β can expressed in the following forms: $\beta = 1/(1 - v^2/V^2)^{1/2}$ or $\beta = [V^2/(V^2 - v^2)]^{1/2}$ or $\beta = V/(V^2 - v^2)^{1/2}$. We begin with $\tau = a[t - v/(V^2 + v^2) \cdot x']$ and since $x' = x - vt$, we have $\tau = a[t - v(x - vt)/(V^2 + v^2)]$. Next, we find a common denominator for the terms inside the parentheses [. . .], and we find the terms vt^2 and $-vt^2$ cancel out giving us $\tau = a[(V^2 t - vx)/(V^2 + v^2)]$. Now, in an subtle manipulation, we manufacture a common factor of $V^2/(V^2 + v^2)$, which gives us $\tau = a[V^2/(V^2 + v^2) \cdot \{t - (vx)/V^2\}]$. Next, we rewrite the two terms inside {. . . .} as a single fraction, which is $[(tV^2 - vx)/V^2]$, and then we isolate the common factor on the right side of the equation, which gives us $(\tau/a) \cdot [V^2/(tV^2 - vx)] = V^2/(V^2 - v^2)$. Now, we incorrectly take the square root of only the right side of the equation, which gives us $(\tau/a) \cdot [V^2/(tV^2 - vx)] = [V^2/(V^2 - v^2)]^{1/2}$. Next, we isolate the term τ on the left side of the equation, and replace the second form of β, $[V^2/(V^2 - v^2)]^{1/2}$, with the third form of β, $V/(V^2 - v^2)^{1/2}$, which gives us $\tau = a[V/(V^2 + v^2)^{1/2} \cdot (t - vx/V^2)]$. It should be noted the two terms inside {. . . .} that were rewritten as the single fraction $[(tV^2 - vx)/V^2]$ now resume their original form as the two terms $(t - vx/V^2)$. Now, since $a = \varphi(v)$ and $\beta = V/(V^2 - v^2)^{1/2}$, we generate Einstein's Eq. (3.11), $\tau = \varphi(v)\beta(t - vx/V^2)$.

Einstein makes the same mistake with Eq. (3.12), $\xi = \varphi(v)\beta(x - vt)$. Eq. (3.12) is the second transformation equation. It transforms the spacial coordinate x of the rest system K into the spacial coordinate ξ of the moving system k. Again, Einstein's mistake is that he takes the square root of one side of the equation but not the other. First, we should explain the derivation of Eq. (3.12). Einstein states, "For a light ray emitted at the time $\tau = 0$ in the direction of increasing ξ, we have Eqs. (3.5), $\xi = V\tau$, or $\xi = aV[t - v/(V^2 - v^2) \cdot x']$. But as measured in the rest system, the light ray propagates with velocity $V - v$ relative to the origin of k, so that $x'/(V - v) = t$. Substituting this value of t in the equation for ξ, we obtain Eq. (3.7), $\xi = a[V^2/(V^2 - v^2)] \cdot x'$." [56] This much is correct, but note that the term $V^2/(V^2 - v^2)$ it is equal to β^2 not β. Einstein concludes by claiming that if we substitute $(x - vt)$ for x' we will obtain $\xi = \varphi(v)\beta(x - vt)$, but that is incorrect.

A more detailed description of the erroneous calculation will be helpful. Eq. (3.5), $\xi = V\tau$, is a version of the equation distance equals velocity multiplied by time. Since $\tau = a[t - v/(V^2 - v^2) \cdot x']$, the equation $\xi = V\tau$ can be rewritten as $\xi = aV[t - v/(V^2 - v^2) \cdot x']$. The term t in the equation $\xi = aV[t - v/(V^2 - v^2) \cdot x']$ can be replaced with $x'/(V - v)$ because, as measured from the rest system, the light beam travels with velocity $(V - v)$ relative to the origin of k, and the distance it travels is x' so that $x'/(V - v) = t$, Eq. (3.6). Substituting this value of t in the equation $\xi = aV[t - v/(V^2 - v^2) \cdot x']$ gives us the following: $\xi = a[V^2/(V^2 - v^2)] \cdot x'$.

The substitution of the left side of Eq. (3.6), which is $x'/(V - v)$, for the term t in Eq. (3.5) is strikingly fortuitous because of the way it simplifies the equation. The denominator of Eq. (3.6), $(V - v)$, is one of the two factors of the denominator of Eq. (3.5), $(V^2 - v^2)$. The other factor is $(V + v)$. With the substitution for t, Eq. (3.5) becomes $\xi = aV[x'/(V - v) - v/(V^2 - v^2) \cdot x']$. The equation with a common denominator for the terms inside the parentheses [. . . .] is $\xi = aV[Vx' + vx' - vx']/(V^2 - v^2)$. The terms $+vx'$ and $-vx'$ cancel out and multiplying V times V gives us $\xi = a[V^2/(V^2 - v^2) \cdot x']$. To obtain Eq. (3.12) we need to isolate the term $V^2/(V^2 - v^2)$ on one side of the equation because the term $V^2/(V^2 - v^2)$ is equal to β^2. Next, we incorrectly take the square root of only that side of the equation, which yields $\xi/(ax') = [V^2/(V^2 - v^2)]^{1/2} = V/(V^2 - v^2)^{1/2}$. Now, we isolate ξ on one side of the equation, and since $a = \varphi(v)$ and $\beta = V/(V^2 - v^2)^{1/2}$ and $x' = x - vt$, we obtain Eq. (3.12), $\xi = \varphi(v)\beta(x - vt)$.

Einstein makes a mistake that involves the term β in both Eq. (3.13), $\eta = \varphi(v)y$, and Eq. (3.14), $\zeta = \varphi(v)z$. Eqs. (3.13) and (3.14) are the third and fourth transformation equations respectively. Eq. (3.13) transforms the y coordinates of the rest system K into the η coordinates of the moving system k. Eq. (3.14) transforms the z coordinates of the rest system K into the ζ coordinates of the moving system k.

Einstein's mistake is that he completely ignores the term β. Also he introduces two special conditions: $x' = 0$ and $t = y/(V^2 - v^2)^{1/2}$. The equation $x' = 0$ limits the moving system k so that it travels no distance at all or zero distance. Since moving system k is moving with a constant velocity along the X-axis of the rest system K, to travel zero distance would take zero time. In zero time the light beam traveling along the η-axis would travel zero distance. This is similar to the situation presented in Eq. (3.1); the light beam makes a round-trip journey that is infinitesimally small–the distance it travels is zero. The equation $t = y/(V^2 - v^2)^{1/2}$ is version of the equation time = distance/speed. According to

Einstein, as measured from the rest system K, the height to which the light beam rises is y. We should recall Einstein states that a light beam traveling along the η-axis in the direction of increasing η in the moving system k will be seen by observers in the rest system K to have originated in the rest system K and to travel along an upward moving diagonal path. Thus, according to Einstein, the measure of the (component of ?) velocity that the light beam displays along the η-axis as observed from the rest system K is $(V^2 - v^2)^{1/2}$.

It will be helpful to explain the derivation of Eqs. (3.13) and (3.14). Einstein refers to Eq. (3.5), $\xi = V\tau$ and states, "Analogously, by considering light rays moving along the two other axes, we get Eq. (3.8), $\eta = V\tau = aV[t - v/(V^2 - v^2) \cdot x']$, where $y/(V^2 - v^2)^{1/2} = t$ and $x' = 0$; hence [we obtain] Eqs. (3.10), $\eta = a[V/(V^2 - v^2)^{1/2} \cdot y]$ and $\zeta = a[V/(V^2 - v^2)^{1/2} \cdot z]$."[57] Since Einstein has stated that $x' = 0$ the final term in Eq. (3.8), $v/(V^2 - v^2) \cdot x'$, is eliminated because under Einstein's conditions it equals zero. We are left with the first term $aV[\, t\,]$, and since $t = y/(V^2 - v^2)^{1/2}$ we have the following $\eta = a[\, V/(V^2 - v^2)^{1/2} \cdot y]$. To obtain Eq. (3.13), $\eta = \varphi(v)y$, from the previous equation we must completely ignore the term $V/(V^2 - v^2)^{1/2}$, which is equivalent to β, and since $a = \varphi(v)$, we obtain Eq. (3.13), $\eta = \varphi(v)y$. To obtain Eq. (3.14), $\zeta = \varphi(v)z$, we follow the same procedure. Of course, in this instance $t = z/(V^2 - v^2)^{1/2}$.

Arthur I. Miller acknowledges that the four equations we have discussed, Eqs. (3.11), (3.12), (3.13) and (3.14) cannot be generated in the manner Einstein describes, but he does not go so far as to call it a mistake. Arthur I. Miller's explanation is, "Then, without prior warning Einstein replaced $a(v)$ with $\varphi(v)(1 -$

$v^2/V^2)^{½}$ to obtain Eqs. (3.11), (3.12), (3.13) and (3.14)."[58] It should be noted for clarity that Einstein denoted a(v) as merely a and with only a few exceptions Miller follows Einstein's example. Miller's explanation does explain away Einstein's mistakes, but it is odd that Einstein did not introduce this replacement of a with $\varphi(v)(1 - v^2/V^2)^{½}$ himself.

Most likely there is a good reason why Einstein never introduced the notion that suddenly a could be replaced with $\varphi(v)(1 - v^2/V^2)^{½}$. Einstein states that a is an unknown function φ(v). All indications are that a was obtained by integration. A quantity obtained by integration cannot shift from φ(v) to $\varphi(v)(1 - v^2/V^2)^{½}$. For example, $\int 2x = x^2$, and we cannot suddenly claim $\int 2x = 2x^2$ because $\int 4x = 2x^2$. Thus, if $\int \partial\tau/\partial x' = \int - v/(V^2 + v^2)\partial\tau/\partial t$ produces $\tau = a[t - v/(V^2 + v^2) \cdot x']$ it cannot also produce $\tau = a\,(1 - v^2/V^2)^{½}\,[t - v/(V^2 + v^2) \cdot x']$.

There is another reason why Einstein never stated that a should be replaced with $\varphi(v)(1 - v^2/V^2)^{½}$. In the final paragraph of the third section Einstein states that φ(v) = 1, and thus the term φ(v) disappears from the transformation equations in their final form. At the beginning of the third section of Einstein's paper a is introduced as an unknown function φ(v), and at the conclusion of the third section of his paper φ(v) is shown to equal one. Therefore, it is inconceivable that a or φ(v) or 1 should ever be replaced with $\varphi(v)(1 - v^2/V^2)^{½}$.

As we have noted, Einstein refers to the four equations we have discussed as transformation equations, and he uses them to demonstrate that there is no contradiction between a constant velocity of light and the principle of relativity. To demonstrate there is no contradiction between a constant velocity of light and the principle of relativity he introduces the notion of a spherical wave. An example of a spherical wave of light is the light produced from a match, a candle or the sun.

> Now we have to prove that, measured in the moving system, every light ray propagates with the velocity V, if it does so, as we have assumed, in the rest system; for we have not yet proved that the principle of the constancy of the velocity of light is compatible with the relativity principle.
> Suppose that at time $t = \tau = 0$ a spherical wave is emitted from the coordinate origin, which at that time is common to both systems, and that this wave propagates in the system K with the velocity V. Hence, if (x, y, z) is a point reached by this wave, we have $x^2 + y^2 + z^2 = V^2t^2$. Eq. (3.16)
> We transform this equation using our transformation equations and, after a simple calculation, obtain
> $\xi^2 + \eta^2 + \zeta^2 = V^2\tau^2$. Eq. (3.17)
> Thus, our wave is also a spherical wave with propagation velocity V when it is observed in the moving system. This proves that our two fundamental principles are compatible.[59]

Einstein does not provide us with the details of this “simple calculation” in his paper On the Electrodynamics of Moving Bodies. But, in “Appendix One” of his book Relativity: the Special and the General Theory he does provide a version of this “simple calculation” that is seven pages in length. One of the errors that invalidates his calculations occurs on the second page so it is worthwhile to reproduce his calculations up to that point.

Simple Derivation of the Lorentz Transformation

> For the relative orientation of the co-ordinate systems indicated in Fig. 2, the x-axes of both systems permanently coincide. In the present case we can divide the problem into parts by considering first only events which are localized on the x-axis. Any such event is represented with respect to the co-ordinate system K by the abscissa x and the time t, and with respect to the system K' by the abscissa x' and t' when x and t are given.
> A light-signal, which is proceeding along the positive axis of x, is transmitted according to the equation $x = ct$ or
> $x - ct = 0$ . . . . (1).
> Since the same light-signal has to be transmitted relative to K' with the velocity c, the propagation relative to the system K' will be represented by the analogous formula
> $x' - ct' = 0$ . . . . (2).
> Those space-time points (events) which satisfy (1) must also satisfy (2). Obviously this will be the case when the relation
> $(x' - ct') = \lambda(x - ct)$. . . . (3) is fulfilled in general, where λ indicates a constant; for according to (3), the disappearance of $(x - ct)$ involves the disappearance of $(x' - ct')$.
> If we apply quite similar considerations to light rays which are being transmitted along the negative x-axis, we obtain the condition $(x' + ct') = \mu(x + ct)$. . . . (4).[60]

Equation (4) is incorrect. The correct equation is the following: $(-x' + ct') = \mu(-x + ct)$. Since the light rays as Einstein states, "are being transmitted along the negative x-axis " of both the K and K' systems the values for x and x' are $-x$ and $-x'$ respectively. As Einstein states, "A light-signal, which is proceeding along the positive axis of x, is transmitted according to the equation $x = ct$."[61] When a light-signal travels along the positive x-axis, the distance the light-signal travels is expressed by the term $+x$. Therefore, when a light-signal travels along the negative x-axis, the distance the light-signal travels must be expressed by the term $-x$. Thus, in the K system a light-signal traveling along the negative x-axis is transmitted according to the equation $-x = -ct$ or $-x + ct = 0$ or if we multiply both sides of the equation by (-1), then $x - ct = 0$. In the K' system a light-signal traveling along the negative x-axis is transmitted according to the equation $-x' = -ct'$ or $-x' + ct' = 0$. Therefore, formula (4) in its correct version is the following: $(-x' + ct') = \mu(-x + ct)$. This error occurs near the beginning of Einstein's demonstration that the equation $x^2 + y^2 + z^2 = V^2t^2$ is equivalent to the equation $\xi^2 + \eta^2 + \zeta^2 = V^2\tau^2$, and the error invalidates his proof.

Eight paragraphs further on there is a second error. Interestingly, this second error is similar to one we encountered in our analysis of the transformation equations. Instead of taking the square root of one side of an equation but not the other side, now, Einstein's error is that one side of an equation is squared while the other side is not squared.

> But if the snapshot be taken from K' (t' = 0), and if we eliminate t from the equations (5), taking into account the expression (6), we obtain $x' = a(1 - v^2/c^2)x$.[62]

The equations (5) are the following: $x' = ax - bct$ and $ct' = act - bx$. We follow Einstein's instructions by letting $t' = 0$ and letting $t = 0$, as well, since this eliminates t from equations (5). As we just noted, letting $t = 0$ allows us to eliminate t from the equations. This gives us the following: $x' = ax - 0$ and $0 = 0 - bx$. Next, we add the two equations together, and the result is the following: $x' = ax - bx$. Equation (6) is the following: $v = bc/a$, and solving for b gives us $b = av/c$. Thus, we can rewrite $x' = ax - bx$ as $x' = ax - av/c \cdot (x)$ or taking into account the common factor ax the equation becomes $x' = ax(1 - v/c)$. To isolate the term v/c on one side of the equation, we divide each side of the equation by ax, next we subtract 1 from each side of the equation, then we multiply each side of the equation by (−1). Thus, we obtain $1 - x'/ax = v/c$. Next, we square the right side of the equation, but do not square the terms on the other side of the equation, and we obtain$1 - x'/ax = v^2/c^2$. To obtain Einstein's equation $x' = a(1 - v^2/c^2)x$, of course, we need to rewrite the equation so that the term x' is isolated on one side of the equation.

Apart from these two errors Einstein accomplishes his task of demonstrating that the equation $x^2 + y^2 + z^2 = V^2t^2$ is equal to the equation $\xi^2 + \eta^2 + \zeta^2 = V^2\tau^2$. Although, these two errors are significant enough to invalidate his proof.

Lillian R. Lieber tries to accomplish the same task in her book The Einstein Theory of Relativity. She encounters difficulties similar to those of Einstein. She introduces the imaginary number $I = (-1)^{1/2}$ to assist her in her efforts, but it is to no avail. Her terminology is slightly different from the terminology Einstein employed in his paper. She uses the following terms that are different from Einstein's terms: t' instead of τ , x' instead of ξ , c instead of V and Einstein's term φ(v) is entirely absent from her Lorentz transformation equations. (This absence will be fully explained in our analysis of the final portion of the third section of Einstein's paper.) Taking these changes into consideration, her versions of the transformation equations are similar to Einstein's versions. In order for L. R. Lieber to accomplish her task, she must set the speed of light, c, equal to one with apparently no accompanying units of measure. As she states below this is done to introduce "simplicity" into the equations. L. R. Lieber's use of “simplicity” is similar to Einstein’s use of “naturalness” and “logical simplicity.”

> Let us examine the similarity between [equation] (20) and the Lorentz transformation a little more closely, selecting from the Lorentz transformation only those equations involving x and t, and disregarding those containing y and z, since the latter remain unchanged in going from one coordinate system to the other. Thus we wish to compare [equation] (20) with: $x' = \beta (x - vt)$ and $t' = \beta(t - vx/c^2)$. Or, if, for simplicity, we take $c = 1$, that is, taking the distance traveled by light in one second, as the unit of distance, we may say that we wish to compare [equation] (20) with $x' = \beta (x - vt)$ and $t' = \beta(t - vx)$. . . . (21)[63]

The term vx/c^2 from the equation $t' = \beta(t - vx/c^2)$ is a measure of time and this is appropriate because it is subtracted from t, which is a measure of time, as well. For instance, v could be measured in miles/second, x could be measured in miles and c^2 could be measured in miles squared/seconds squared. This gives us the complex fraction (miles squared/second)/(miles squared/seconds squared), which simplifies to seconds, a measure of time. The term vx from equation (21) is not a measure of time. It is a measure of length squared/unit of time. It would not be appropriate to subtract this term from a measure of time. L. Lieber continues her demonstration.

> Let us first solve [equation] (21) for x and t, so as to get them more nearly in the form of [equation] 20. By ordinary algebraic operations, we get the following: $x = \beta(x' + vt')$ and $t = \beta(t' +vx')$ . . . . (22)[64]

There are no ordinary algebraic operations that can produce the two equations above. Solving for x and t we obtain the following: $x = x'/\beta + vt$ and $t = t'/\beta + vx$. Notice that when we solve for x and t respectively the final terms in the respective equations are vt and vx as opposed to vt' and vx'.

L. Lieber next introduces the square root of negative one into the proceedings written in the form $i = (-1)^{1/2}$.

> Let us now return to the comparison of [equations] (22) and (20): Minkowski pointed out that if, in (22), t is replaced by $i\tau$ {where $i = (-1)^{1/2}$}, and t' by $i\tau'$, then (22) becomes $x = \beta x' + i\beta v\tau'$ and $i\tau = i\beta\tau' + \beta vx'$. Or (by multiplying the second equation by $-i$): $x = \beta x' + i\beta v\tau'$ and $\tau = \beta\tau' - i\beta vx'$.[65]

The replacement of t and t' with $i\tau$ and $i\tau'$ respectively is arbitrary. It is a clever way to manufacture formulas where certain inconvenient terms will cancel out, thus allowing the demonstration of the desired point. She is not following the accepted practice of multiplying each side of an equation by the same quantity such as five or seven. Instead she is multiplying a selected term or selected terms of an equation by i . L. Lieber continues with her demonstration.

> Finally, substituting $\cos\theta$ for β and $\sin\theta$ for $-i\beta v$ these equations become $x = x'\cos\theta - \tau'\sin\theta$ and $\tau = x'\sin\theta + \tau'\cos\theta$. . . . (23)[66]

These substitutions are justified by L. Lieber in the following clever manner.

> Note that $\sin^2\theta + \cos^2\theta = 1$ holds for imaginary angles as well as for real ones; hence the above substitutions are legitimate, thus $\beta^2 + (-i\beta v)^2 = \beta^2 - \beta^2 v^2 = \beta^2(1 - v^2) = 1$ since $\beta^2 = 1/(1 - v^2)$, c being taken equal to one.[67]

The mathematical sequence above can be confusing. It is helpful to rewrite the term $(-i\beta v)^2$ as $(-1i\beta v)^2$ or $(-1)^2 i^2 \beta^2 v^2$. Since $(-1)^2 = 1$ and $i^2 = -1$ the term $(-i\beta v)^2$ is correctly presented as $-\beta^2 v^2$.

L. Lieber's next step is to square each side of the two equations denoted as equation (23).

> And since, from [equation] (23)
> $x^2 = (x')^2 \cos^2\theta - 2x'\tau' \sin\theta \cos\theta + (\tau')^2 \sin^2\theta$
> $\tau^2 = (x')^2 \sin^2\theta + 2x'\tau' \sin\theta \cos\theta + (\tau')^2 \cos^2\theta$
> then, obviously,
> $x^2 + \tau^2 = (x')^2 + (\tau')^2$
> or (since $y = y'$ and $z = z'$),
> $x^2 + y^2 + z^2 + \tau^2 = (x')^2 + (y')^2 + (z')^2 + (\tau')^2$.[68]

When squared versions of equations (23) are added together, it is important to recall that $\sin^2\theta + \cos^2\theta = 1$.

Despite L. Lieber's best efforts, she does not succeed in demonstrating that $x^2 + y^2 + z^2 + \tau^2 = (x')^2 + (y')^2 + (z')^2 + (\tau')^2$. She fails to explain convincingly how c^2 may be replaced the number one, especially since it is the number one without any units of measure attached to it. Also, she fails to explain the algebraic operations she employs to obtain vt' and vx' from equations that contain vt and vx respectively. Finally, she does not justify the replacement of τ and τ' with $i\tau$ and $i\tau'$ respectively.

We can now turn our attention to the final portion of the third section. In this portion of the third section, Einstein determines that the function $\varphi(v) = 1$. His method has a certain arbitrariness to it. He introduces a third coordinate system denoted as K' that turns out to be at rest relative to the rest system K.

> The transformation equations we have derived also contain an unknown function φ of v, which we now wish to determine.
> To this end we introduce a third coordinate system K', which, relative to the system k, is in parallel-translational motion, parallel to the axis Ξ, such that its origin moves along the Ξ-axis with velocity −v. Let all three coordinate origins coincide at time t = 0, and let the time t' of system K' equal zero at t = x = y = z = 0. We denote the coordinates measured in the system K' by x', y', z' and by twofold application of our transformation equations, we get
>
> $$t' = \varphi(-v)\beta\,[\tau + v/V^2 \cdot \xi] = \varphi(v)\varphi(-v)t, \qquad (3.18)$$
>
> $$x' = \varphi(-v)\beta\,[\xi + v\tau] = \varphi(v)\varphi(-v)x, \qquad (3.19)$$
>
> $$y' = \varphi(-v)\eta = \varphi(v)\varphi(-v)y, \qquad (3.20)$$
>
> $$z' = \varphi(-v)\zeta = \varphi(v)\varphi(-v)z. \qquad (3.21)$$ [69]

In the Greek alphabet "Ξ" is the upper case representation of the English letter "x." As Einstein states the Eqs. (3.18)–(3.21) are produced by a "twofold application of our transformation equations." To produce the equations on the far right the values of τ, ξ, η, and ζ are represented in the coordinates of the rest system K as determined by the transformation equations. Then these equations are manipulated to produce the equations on the far right.

The procedure can be confusing so it is useful to go through the intermediate steps. We begin with $t' = \varphi(-v)\beta\,[\tau + v/V^2 \cdot \xi]$ and then we substitute the rest system K values for τ and ξ, which gives us the following: $t' = \varphi(-v)\beta\,\{\varphi(v)\beta\,(t - v/V^2 \cdot x) + v/V^2\,[\varphi(v)\beta\,(x - vt)]\}$. The term $\varphi(v)\beta$ is a common factor for the terms inside {. . . .} so we can rewrite the equation as follows: $t' = \varphi(-v)\beta\,\{\varphi(v)\beta\,[t - v/V^2 \cdot x + v/V^2 \cdot x - v^2/V^2 \cdot t]\}$. We note that the terms $-v/V^2 \cdot x$ and $+ v/V^2 \cdot x$ cancel out and the two remaining terms inside the parentheses [. . .] have t as a common factor and thus can be expressed as $t(1 - v^2/V^2)$. This gives us $t' = \varphi(-v)\beta \{ \varphi(v)\beta\, t(1 - v^2/V^2)\}$. Now we recall that $\beta = 1/(1 - v^2/V^2)^{½}$ so we can rewrite the equation as the following: $t' = \varphi(-v)\, \varphi(v)\, t \cdot 1/(1 - v^2/V^2)^{½} \cdot 1/(1 - v^2/V^2)^{½} \cdot (1 - v^2/V^2)$. The last three terms cancel out because in the denominators of the first and second terms in question we have the square root of a quantity times the square root of the same quantity and the third term in question consists of the quantity itself. For example, $1/(25)^{½} \bullet 1/(25)^{½} \bullet 25 = 1$. Thus we have $t' = \varphi(-v)\, \varphi(v)\, t$.

The procedure is the same for the equation $x' = \varphi(-v)\beta\, (\xi + v\tau)$ so we will list the five steps without any commentary.

$x' = \varphi(-v)\beta\, [\, \varphi(v)\beta\, (x - vt) + v(\varphi(v)\beta\, (t - v/V^2 \cdot x)]$

$x' = \varphi(-v)\beta\, [\, \varphi(v)\beta\, (x - vt + vt - v^2/V^2 \cdot x)]$

$x' = \varphi(-v)\beta\ \varphi(v)\beta\, x\, (1 - v^2/V^2)$

$x' = \varphi(-v)\ \varphi(v)\, x \cdot 1/(1 - v^2/V^2)^{½} \cdot 1/(1 - v^2/V^2)^{½} \cdot (1 - v^2/V^2)$

$x' = \varphi(-v)\ \varphi(v)\, x$

In order to transform the equation $y' = \varphi(-v)\eta$, we recall that $\eta = \varphi(v)y$, which gives us the following: $y' = \varphi(v)\, \varphi(-v)y$. In order to transform the equation $z' = \varphi(-v)\zeta$, we recall that $\zeta = \varphi(v)z$, which gives us the following: $z' = \varphi(v)\, \varphi(-v)z$.

Following the equations above Einstein informs us that $\varphi(v) \cdot \varphi(-v) = 1$. This may be true for the special circumstances engendered by the fact that the system K' is at rest relative to the system K, but is it universally true?

> Since the relations between x', y', z' and x, y, z do not contain the time t, the systems K and K' are at rest relative to each other, and it is clear that the transformation from K to K' must be the identity transformation. Hence, $\varphi(v) \cdot \varphi(-v) = 1$. (3.22)[70]

Would $\varphi(v) \cdot \varphi(-v) = 1$ if the systems K and K' were not at rest relative to each other? Since the transformation from K to K' is an identity transformation, it would be more accurate to state that under special circumstances $\varphi(v) \cdot \varphi(-v) = 1$. Einstein continues his investigation of the function $\varphi(v)$. It should be pointed out that the letter "H" is the upper case version of the letter "η" in the Greek alphabet.

> Let us now explore the meaning of $\varphi(v)$. We shall focus on that portion of the H-axis of the system k that lies between $\xi = 0, \eta = 0, \zeta = 0$, and $\xi = 0, \eta =$ length l, $\zeta = 0$. This portion of the H-axis is a rod that, relative to the system K, moves perpendicular to its axis with a velocity v and its ends have the coordinates in K:
>
> $$x_1 = vt, \quad y_1 = l/\varphi(v), \quad z_1 = 0 \qquad (3.23)$$
>
> and
>
> $$x_2 = vt, \quad y_2 = 0, \quad z_2 = 0. \qquad (3.24)$$
>
> The length of the rod, measured in K, is thus $l/\varphi(v)$; this gives us the meaning of the function φ. . . . Thus, the length of the moving rod measured in the rest system does not change if v is replaced by −v. From this we conclude:
>
> $$l/\varphi(v) = l/\varphi(-v) \quad \text{or} \quad \varphi(v) = \varphi(-v). \qquad (3.25)$$
>
> From this relation and the one found earlier it follows that $\varphi(v) = 1$. . ..[71]

The explanation for the formula $y_1 = l/\varphi(v)$ is that l is substituted for η in the transformation formula $\eta = \varphi(v)y$ thus becoming $l = \varphi(v)y$, which can be rewritten as $y_1 = l/\varphi(v)$. The explanations for the formulas $x_1 = vt$ and $x_2 = vt$ are that 0 is substituted for ξ in the transformation formula $\xi = \varphi(v)\beta(x - vt)$. Zero divided by $\varphi(v)\beta$ is zero, which gives us $0 = x - vt$ or $x = vt$.

The result of finding the meaning of the function $\varphi(v)$ is that since, according to Einstein, it equals one, the function can be eliminated from the transformation equations. Thus the absence of the term $\varphi(v)$ in L. Lieber's version of the transformation equations is explained.

Since $\varphi(v) = 1$, the transformation equations take the following form:

$$\tau = \beta(t - vx/V^2), \quad (3.26)$$

$$\xi = \beta(x - vt), \quad (3.27)$$

$$\eta = y, \quad (3.28)$$

$$\zeta = z, \quad (3.29)$$

where $\beta = 1/[1 - (v^2/V^2)]^{1/2}$.[72]

Thus, the third section ends with the listing of the transformation equations in their final form.

Part Three

An Analysis of Section 4. The Physical Meaning of the Equations Obtained as Concerns Moving Rigid Bodies

and Moving Clocks

In the fourth section Einstein demonstrates two concepts. The first concept is the contraction of matter along the axis aligned with the direction of motion of an object. The second concept is the dilation or slowing down of time for an object in motion.

The object he uses in his first demonstration is a sphere with radius R. This sphere is at rest relative to the moving system k, and the center of the sphere is located at the origin of k. The equation for the surface of the sphere is $\xi^2 + \eta^2 + \zeta^2 = R^2$ Eq. (4.1).

An example will help us understand this equation. Let $\xi = \eta = \zeta = 4$ feet. We start at the origin of k, which is also the center of the sphere and move four feet to the right along the ξ-axis. Next, we move four feet upward (in the ξ–η plane) in a direction perpendicular to the ξ-axis. Using the Pythagorean theorem ($a^2 + b^2 = c^2$) we calculate we are $(32)^{1/2}$ feet from the origin because 4^2 sq. ft.+ 4^2 sq. ft. = 32 sq. ft. The four foot lengths form the sides of a right triangle and $(32 \text{ sq. ft.})^{1/2}$ is the length of the hypotenuse. Next, we move perpendicularly four feet above the ξ–η plane. We have reached a point on the surface of the sphere. We can calculate the radius of the sphere using the Pythagorean theorem. The four feet we moved above the ξ–η plane is one leg of a right triangle. The other leg of the right triangle is the distance $(32)^{1/2}$ feet, which is the distance we moved from the origin in the ξ–η plane. Thus 4^2 sq. ft. + $[(32 \text{ sq. ft.})^{1/2}]^2 = 48$ sq. ft. and the radius of the sphere is $(48 \text{ sq. ft.})^{1/2}$ or about seven feet.

This sphere of about seven feet in radius is at rest relative to the moving system k. Considered from the rest system K the sphere's dimensions are expressed in terms of x, y, and z. Einstein states, "Expressed in terms of x, y, and z, the equation of this surface at time t = 0 is $x^2/\{[(1 - v^2/V^2)]^{1/2}\}^2 + y^2 + z^2 = R^2$ Eq. (4.2)"[73] The first term of Eq. (4.2), $x^2/\{[(1 - v^2/V^2)]^{1/2}\}^2$, is obtained by using the equation $\xi = \beta(x - vt)$ when t = 0, which gives us the following: $\xi = \beta(x)$ and therefore $\xi^2 = \beta^2(x)^2$. Since β is equal to $1/(1 - v^2/V^2)^{1/2}$, we obtain $\xi^2 = x^2/\{[(1 - v^2/V^2)]^{1/2}\}^2$, which can be simplified as follows: $\xi^2 = x^2/(1 - v^2/V^2)$ or $\xi = x/(1 - v^2/V^2)^{1/2}$.

If we return to our example where $\xi = \eta = \zeta = 4$ feet, we can use the transformation equations $\eta = y$ and $\zeta = z$ to calculate that y = 4 feet and z = 4 feet. But, we cannot let x = 4 feet because that would result in ξ being equal to 4 feet$/(1 - v^2/V^2)^{1/2}$, which would result in a value for ξ larger than 4 feet for velocities (v) greater than zero and less than the velocity of light. If we let x = 4 feet • $(1 - v^2/V^2)^{1/2}$, the resulting value of ξ is 4 feet. To summarize the procedure, we know that x must be divided by $(1 - v^2/V^2)^{1/2}$ and this result must equal ξ, which in our example is 4 feet. We also know that when x is divided by $(1 - v^2/V^2)^{1/2}$ the result will be larger than x when the velocity (v) is greater than zero and less than the velocity of light. This is because x will be divided by a fraction greater than zero and less than one. For example if, $(v^2/V^2) = 3/4$, the result is $(1 - 3/4)^{1/2}$, which is equal to $(1/4)^{1/2}$ or ½. When x is divided by ½ the result is 2x. To counter this we must multiply x by ½ or $(1 - 3/4)^{½}$. In general we must let $x_{\text{for calculation purposes only}} = \xi$ and multiply $x_{\text{for calculation purposes only}}$ by $(1 - v^2/V^2)^{½}$ to obtain x.

When $\xi = R$ where R is the radius of the sphere, Einstein states, "A rigid body that has a spherical shape when measured at rest has, when in motion—considered from the rest system— the shape of an ellipsoid of revolution with axes $R(1 - v^2/V^2)^{½}$, R, R. Eq. (4.3)"[74]

Now, Einstein turns his attention to his second demonstration, which is the slowing down or dilation of time for objects in motion.

> We further imagine one of the clocks that is able to indicate time t when at rest relative to the rest system and time τ when at rest relative to the moving system to be placed at the origin of k and set such that it indicates the time τ. What is the rate of this clock when considered from the rest system?
>
> The quantities x, t, and τ that refer to the position of this clock obviously satisfy the equations
> $\tau = 1/(1 - v^2/V^2)^{½} \cdot [t - (v/V^2 \cdot x)]$ (4.4)
> and
> $x = vt.$ (4.5)
> We thus have
> $\tau = t(1 - v^2/V^2)^{½} = t - [1 - (1 - v^2/V^2)^{½}] \cdot t,$ (4.6)
> from which it follows that the reading of the clock considered from the rest system lags behind each second by $[1 - (1 - v^2/V^2)^{½}]$ sec. or, up to quantities of the fourth and higher order, by $½(v^2/V^2)$ sec.
>
> This yields the following peculiar consequence: If at the points A and B of K there are clocks at rest that, considered from the rest system, are running synchronously, and if the clock at A is transported to B along the connecting line with velocity v, then upon arrival of this clock at B the two clocks will no longer

> be running synchronously; instead, the clock that has been transported from A to B will lag ½ $t(v^2/V^2)$ sec. (up to quantities of the fourth and higher orders) behind the clock that has been in B from the outset where t is the time needed by the clock to travel from A to B. . ..
>
> From this we conclude that a balance-wheel clock located at the Earth's equator must, under otherwise identical conditions, run more slowly by a very small amount than an absolutely identical clock located at one of the Earth's poles.[75]

It will be helpful to show the intermediate steps of the above calculations beginning with Eq. (4.4). If we let $x = vt$ in Eq. (4.4), $\tau = 1/(1 - v^2/V^2)^{1/2} \cdot [t - (v/V^2 \cdot x)]$, the result is the following: $\tau = 1/(1 - v^2/V^2)^{1/2} \cdot [t - (v/V^2 \cdot vt)]$. The final term, $[t - (v/V^2 \cdot vt)]$ has t as a common factor and therefore can be rewritten as $t(1 - v^2/V^2)$. In its rewritten form we can see it shares a common factor with its companion term $1/(1 - v^2/V^2)^{1/2}$. When we divide $(1 - v^2/V^2)$ by its square root $(1 - v^2/V^2)^{1/2}$ the result is $(1 - v^2/V^2)^{1/2}$. Thus we obtain the equation $\tau = t(1 - v^2/V^2)^{1/2}$. We can rewrite the equation as $\tau = t - [1 - (1 - v^2/V^2)^{1/2}] \cdot t$ because the +t and the −t cancel out each other, as do the negative (minus) signs before [. . . .] and (. . . . .).

We should note that the equation $x = vt$ is a special case of the equation $x = x' + vt$ in which x' is equal to zero.

We should also concern ourselves with the meaning of the phrase, "up to quantities of the fourth and higher order." The phrase is used twice, and its meaning is unclear. It seems to imply that for everyone of the individual quantities generated by the use of a particular velocity (v) in the term $[1 - (1 - v^2/$

$V^2)^{1/2}$] sec., the same quantity can be generated by using the same particular velocity (v) in the term $\frac{1}{2}(v^2/V^2)$ sec. This is not the case. The term $\frac{1}{2}(v^2/V^2)$ sec. is an approximation of the term $[1 - (1 - v^2/V^2)^{1/2}]$ sec. This can be demonstrated by the following example. Let $(v^2/V^2) = 19/100$, which gives us $[1 - (1 - 19/100)^{1/2}]$ sec. = $[1 - (81/100)^{1/2}]$ sec. = $(1 - 9/10)$ sec. = 1/10 sec. The term $\frac{1}{2}(v^2/V^2)$ sec. = $\frac{1}{2}(19/100)$ sec. = 19/200 sec., which is 1/200 sec. less than 1/10 sec.

As we noted the phrase occurs twice, and A. I. Miller's first translation of the phrase is, "neglecting magnitudes of fourth and higher order,"[76] and his second translation of the phrase is, "up to magnitudes of fourth and higher order."[77] These two different translations only add to the ambiguity of the phrase.

A. I. Miller also offers a brief explanation of the term $\frac{1}{2}t(v^2/V^2)$. The term $\frac{1}{2}(v^2/V^2)$ is multiplied by t because for each second that passes the clock lags behind $\frac{1}{2}(v^2/V^2)$ seconds when considered from the rest system. Therefore if t seconds pass the clocks lags behind $\frac{1}{2}t(v^2/V^2)$ when considered from the rest system. A. I. Miller's explanation of the term $\frac{1}{2}t(v^2/V^2)$ is, at least, consistent with explanation of Eq. (3.2). It is not correct, but it is consistent. He does not conclude that $\frac{1}{2}t(v^2/V^2)$ is an approximation, and instead he invokes an explanation remarkably similar to his explanation of Eq. (3.2). Miller states, "Expanding the square root in the equation $[1 - (1 - v^2/V^2)^{1/2}]$, and retaining terms up to second order in v/V, gives $\frac{1}{2}t(v^2/V^2)$."[78] We should note that the expression $[1 - (1 - v^2/V^2)^{1/2}]$, which according to Miller, produces the term $\frac{1}{2}t(v^2/$

V^2) when its square root is expanded does not include the variable t, which is necessary for the term ½ $t(v^2/V^2)^{½}$. The absence of the variable t from the expression $[1 - (1 - v^2/V^2)^{½}]$ is not a debilitating condition regarding A. I. Miller's explanation of the term ½$t(v^2/V^2)$ because the variable t can be inferred from the expression's context, i.e., Einstein's statement, "the rest system lags behind each second by $[1 - (1 - v^2/V^2)^{½}]$ . . .."

In the third paragraph Einstein discusses two clocks A and B located at points A and B respectively in the rest system K. These two clocks are running synchronously. When clock A is transported to the location of clock B the two clocks are no longer running synchronously. Clock A will be running behind clock B by the amount of ½ $t(v^2/V^2)$ sec. where t is the time it takes to transport the clock from point A to point B and v is the velocity with which the clock is transported. It is interesting to note that Einstein sees no difficulty in applying an argument developed for clocks in the rest system K to clocks located on the Earth, a moving system.

With that in mind, we will construct a thought experiment and apply his arguments to a rod moving with the velocity of .5c. The rod is 6 light seconds in length or 1,116,000 miles. The rod is moving such that point B is in the forward position and point A is in the rear position. There is a clock at point A that is synchronized with the clock at point B. Clock A is transported to the location of clock B with a velocity of 100,000 miles/second relative to the rod. The time it takes to transport clock A to clock B is 11.16 seconds. The amount of time clock A should lag behind clock B after clock A is transported to clock B is given by the term ½ $t(v^2/V^2)$ sec., which is equal to ½(11.16 sec.) • $100{,}000^2_{(\text{miles/ second})^2}$ /3.46 × $10^{10}_{(\text{mles/second})^2}$ = 1.6 seconds. The final "sec." occurring in the term ½ $t(v^2/V^2)$ sec. is unnecessary. According to our calculations when clock A is transported to clock B, clock A will lag 1.6 seconds behind clock B.

This result implies that distant synchronized clocks agree in the time they keep. As we have seen according to Einstein's definition of distant synchronized clocks, which is $t_B - t_A = t_{A'} - t_B$, these clocks do not have to agree in the time they keep in order to be considered synchronized. Einstein's definition states that the time it takes a light beam to travel from clock A to clock B is equal to the time it takes a light beam to travel from clock B to clock A. Since the velocity of our rod is .5c in a direction that makes point B the forward point, the effective velocity of a light beam traveling from point A to point B is $186{,}000_{(\text{miles/second})} - 93{,}000_{(\text{miles/second})} = 93{,}000_{(\text{miles/second})}$. Therefore, a light beam takes 12 seconds to travels from point A to point B. The effective velocity of a light beam traveling from point B to point A is $186{,}000_{(\text{miles/second})}$ + $93{,}000_{(\text{miles/second})} = 279{,}000_{(\text{miles/second})}$. Therefore, a light beam takes 4 seconds to travel from point B to point A. If the clock at point B is set 4 seconds slow, the clocks will be synchronized according to Einstein's definition. The light beam begins its journey from point A at t_A = 00:00, and it strikes the mirror at point B at t_B = 00:08, of course, at this same instant the clock at point A reads 00:12. The light beam returns to point A, and the clock at point A reads $t_{A'}$ = 00:16. Thus, $t_B - t_A = t_{A'} - t_B$ since 00:08 – 00:00 = 00:16 – 00:08.

Since clock B is 4 seconds behind clock A, when clock A is transported to clock B and the clocks are placed side by side, clock B will be only 2.4 seconds behind clock A instead 4 seconds, because during its journey clock A slows by 1.6 seconds. But, according to Einstein's determination clock A should lag behind clock B by 1.6 seconds. Instead, clock B lags behind clock A by 2.4 seconds. From this discrepancy, it can be determined that clock B runs 4 seconds behind clock A. With this information the absolute velocity of the rod could be determined. Of course, since it is a thought experiment we already know the absolute velocity of the rod is .5c in a direction such that point B is the forward point.

The calculation of the velocity of the rod begins with the fact that it takes the light beam 12 seconds to travel from point A to point B, instead of the 8 seconds the observers originally thought necessary for the first leg of the round-trip journey. The distance the light beam travels is $12_{seconds} \times 186{,}000_{miles/second} = 2{,}232{,}000_{miles}$. If we subtract the length of the rod, we are left with the distance point B rushed away from the oncoming light beam in 12 seconds the calculation is $2{,}232{,}000_{miles} - 1{,}116{,}000_{miles} = 1{,}116{,}000_{miles}$. The distance point B rushed away from the oncoming light beam is 1,116,000 miles. If we divide 1,116,000 miles by the 12 seconds it took the rod to travel this distance we calculate the speed of the rod, which is 93,000 miles/second. Since we know the rod is traveling in a direction that makes point B the forward position, we have calculated the velocity of the rod.

Part Four

An Analysis of Section 5.

The Addition Theorem for Velocities

The mathematics in the fifth section are quite involved, and a thorough explanation of intermediate steps that occur between each equation will prove helpful. We will begin our examination with the two paragraphs with which Einstein begins this section.

> In the system k moving with velocity v along the X-axis of the system K, let a point move in accordance with the equations
>
> $$\xi = \omega_\xi \tau \qquad (5.1)$$
> $$\eta = \omega_\eta \tau \qquad (5.2)$$
> $$\zeta = 0, \qquad (5.3)$$
>
> where ω_ξ and ω_η denote constants.
>
> We seek the motion of the point relative to the system K. Introducing the quantities x, y, z, t into the equations of motion of the point by means of the transformation equations derived in section 3, we obtain
>
> $$x = (\omega_\xi + v)/[1 + (v\omega_\xi / V^2)] \cdot t, \qquad (5.4)$$
>
> $$y = [1 - (v^2 / V^2)]^{1/2} / [1 + (v\omega_\xi / V^2)] \cdot \omega_\eta t, \qquad (5.5)$$
>
> $$z = 0. \qquad (5.6)$$ [79]

The Eq. (5.4) is generated from the Eq. (5.1). In Eq. (5.1) the term ξ is replaced with $\beta(x - vt)$ from the transformation equation $\xi = \beta(x - vt)$, and the term τ is replaced with $\beta(t - vx/V^2)$ from the transformation equation $\tau = \beta(t - vx/V^2)$. This gives us $\beta(x - vt) = \omega_\xi[\beta(t - vx/V^2)]$, which can be rewritten as $\beta x - \beta vt = \omega_\xi \beta t - \omega_\xi \beta(vx/V^2)$. If we group the terms with x in them on the left side of the equation and the terms with t in them on the right side of the equation we obtain $\beta x + \omega_\xi\beta(vx/V^2) = \beta t(\omega_\xi + v)$. Then we divide each side of the equation by β, and we obtain the following: $x + \omega_\xi(vx/V^2) = t(\omega_\xi + v)$. Since x is a common factor on the left side of the equation, we can rewrite the equation as the following: $x(1 + \omega_\xi v/V^2) = t(\omega_\xi + v)$. Finally, we divide each side of the equation by $(1 + \omega_\xi v/V^2)$, and we obtain Eq. (5.4), $x = t(\omega_\xi + v)/(1 + \omega_\xi v/V^2)$.

The generation of Eq. (5.5) involves the use of a complex fraction. A complex fraction is a fraction constructed of a fraction in the numerator and a fraction in the denominator. For example, (½)/(¼) is a complex fraction. To simplify a complex fraction we must invert the denominator and then multiply the numerator by the inverted denominator. Thus, the complex fraction (½)/(¼) = (½)• (4/1) = 4/2 = 2. The complex fraction we are going to be involved with is more confusing. In the example above, the denominator of the numerator of the complex fraction is two. In the complex fraction we are going to generate the denominator of the numerator of the complex fraction is itself a fraction. For clarity we will separate the numerator of the complex fraction from the denominator of the complex fraction with the sign ÷, and we also should note the denominator of the complex fraction will be V^2, which we will denote as $V^2/1$.

The Eq. (5.5) is generated from Eq. (5.2). In Eq. (5.2) the term η is replaced with y from the transformation equation $\eta = y$, and the term τ is replaced with $\beta(t - vx/V^2)$ from the transformation equation $\tau = \beta(t - vx/V^2)$. Thus we have $y = \omega_\eta \beta(t - vx/V^2)$. We replace x with $t(\omega_\xi + v)/(1 + \omega_\xi v/V^2)$ from Eq. (5.4), $x = t(\omega_\xi + v)/(1 + \omega_\xi v/V^2)$. This gives us the following: $y = \omega_\eta \beta [t - vt(\omega_\xi + v)/(1 + \omega_\xi v/V^2) \div V^2/1]$. We can rewrite the denominator of the numerator of the complex fraction as the following: $(V^2 + v\omega_\xi)/V^2$. Thus, we obtain $y = \omega_\eta \beta [t - vt(\omega_\xi + v)/\{(V^2 + \omega_\xi v)/V^2\} \div V^2/1]$. When we invert the denominator, $V^2/1$, we obtain $1/V^2$. The numerator of the complex fraction is $vt(\omega_\xi + v)/[(V^2 + \omega_\xi v)/V^2]$. When we multiply the numerator of the complex fraction by the inverted denominator we obtain $[vt(\omega_\xi + v) \bullet 1]/[(V^2 + \omega_\xi v)/V^2] \bullet V^2$. Thus, the compound fraction is simplified, and we obtain $y = \omega_\eta \beta [t - vt(\omega_\xi + v)/(V^2 + \omega_\xi v)]$.

Next, we note that t is a common factor for the terms inside [. . . .]. We can rewrite the equation as $y = \omega_\eta \beta \{t [1 - v(\omega_\xi + v)/(V^2 + \omega_\xi v)]\}$. Now, we find a common denominator for the terms inside [. . . .], which is the following: $(V^2 + \omega_\xi v)$. This allows us to rewrite the terms inside [. . . .] as $(V^2 + \omega_\xi v - \omega_\xi v - v^2/(V^2 + \omega_\xi v)$. We note that the $+\omega_\xi v$ and the $-\omega_\xi v$ cancel out, which leaves us with $(V^2 - v^2)/(V^2 + \omega_\xi v)$. The equation can be rewritten as the following: $y = \omega_\eta \beta t[(V^2 - v^2)/(V^2 + \omega_\xi v)]$. We can rewrite the term $(V^2 - v^2)$ as $V^2(1 - v^2/V^2)$ and we can write β as the following: $1/(1 - v^2/V^2)^{1/2}$. Thus, the equation becomes $y = 1/(1 - v^2/V^2)^{1/2} \bullet V^2(1 - v^2/V^2) \ / (V^2 + \omega_\xi v) \bullet \omega_\eta t$. Since a term is divided by its square root the equation becomes $y = V^2(1 - v^2/V^2)^{1/2}/(V^2 + \omega_\xi v) \cdot \omega_\eta t$. Next we must multiply the numerator and denominator of the fraction by $1/V^2$, which is the same as multiplying by one. The equation becomes the following: $y = (1 - v^2/V^2)^{1/2}/[(V^2 + \omega_\xi v)/V^2] \cdot \omega_\eta t$. This can be rewritten as follows: $y = (1 - v^2/V^2)^{1/2} / (1 + \omega_\xi v/V^2) \cdot \omega_\eta t$, which is Eq. (5.5)

Eq. (5.6), $z = 0$, is generated from Eq. (5.3), $\zeta = 0$ by using the transformation equation $\zeta = z$.

There is an insurmountable difficulty with the reasoning we have employed in the five previous paragraphs. The transformation equation $\xi = \beta(x - vt)$ from which we substituted $\beta(x - vt)$ for ξ in Eq. (5.1), $\xi = \omega_\xi \tau$, is generated from and is in fact equal to the equation $\xi = V\tau$. Thus, we have the impossible situation of $V\tau = \omega_\xi \tau$. The equation states that the velocity of a beam of light, traveling along the ξ-axis in the direction of increasing ξ, multiplied by a specific time interval τ is equal to any constant velocity along the ξ-axis multiplied by the same specific time interval τ.

The equation $\xi = V\tau$ is a special case of the more general equation $\xi = \omega_\xi \tau$ in two fundamental ways. The first way that the equation $\xi = V\tau$ is a special case involves the comparison of the term ω_ξ with the term V. The term ω_ξ represents a component velocity that is the motion of the point as measured along the ξ-axis. That component velocity can assume any particular constant velocity. The term V represents only one specific velocity—the velocity of a light beam—in this case the direction of the light beam is along the ξ-axis in the direction of increasing ξ. The point also has another component velocity ω_η, which represents the velocity of the point's motion as measured along the η-axis, and that component velocity can be any particular constant velocity, as well. The terms ω_ξ and ω_η are the component velocities of a point with a velocity ω. However, the important distinction is that the term V in the equation $\xi = V\tau$ represents only one specific velocity and it is the velocity of a light beam traveling along the ξ-axis in the direction of increasing ξ. The point with velocity ω can have any particular constant velocity. The second way that the equation $\xi = V\tau$ is a special case is that the point with component velocities ω_ξ and ω_η is carried along by the moving system k. The light beam with velocity V is not carried along by the moving system k.

If we think of the moving system k as a long train consisting of many flat-bed cars, which are pulled by a powerful locomotive, the point can be represented by a runner stationed on the last flat-bed car. The runner is carried along by the velocity of the train. If he starts running towards the engine, no matter how great the velocity of the train, he only needs to travel the length of the flat-bed cars to reach the locomotive. However, a light beam projected from the last flat-bed car will not be carried along by the train. The engine will be rushing away from the oncoming light beam, and the light beam will need to travel a distance greater than the length of the flat-cars to reach the locomotive.

To generate the transformation equation $\xi = \beta(x - vt)$, Einstein begins with the equation $\xi = V\tau$, and since $\tau = a[t - vx'/(V^2 - v^2)]$, he generates the equation $\xi = aV [t - vx'/(V^2 - v^2)]$. The term V in bold type represents the velocity of a beam of light. It does not represent the velocity of a point. The transformation equation deals with a light beam with velocity V. The velocity of the light beam becomes combined with other terms so we can easily lose sight of it. There is no reason for equations analyzing the behavior of a point with velocities ω_ξ and ω_η to include the velocity of a beam of light V, but according to Einstein they must.

As we have discussed previously, Einstein continues his analysis of the equation $\xi = aV [t - vx'/(V^2 - v^2)]$. He states, "But as measured in the rest system, the light ray propagates with velocity $V - v$ relative to the origin of K, so that $x'/(V - v) = t$."[80] Since there is no light beam in the thought experiment involving the point with component velocities ω_ξ and ω_η, it is inconceivable that $x'/(V - v) = t$. Instead, the time t can be expressed in three ways: $x'/\omega_\xi = t$, $y/\omega_\eta = t$, or $d/\omega = t$, where d is the distance traveled by the point and ω is the velocity of the point. Since the point is carried along by the moving system k, the point is like the runner running on the flat-bed cars of a train. The velocity of the point and the runner relative to the origin of K is $\omega + v$, not $\omega - v$.

Contrary to Einstein's claim, the equation $\xi = \omega_\xi \tau$ expressed in the quantities x and t assumes the following form: $\xi = a\omega_\xi[(x'/\omega_\xi - v/(V^2 - v^2) \cdot x']$. Since x' is a common factor for the terms within [. . .] and since $x' = x - vt$, the equation becomes $\xi = a\omega_\xi(x - vt)[(1/\omega_\xi - v/(V^2 - v^2)]$. If we replace the variable ω_ξ, which can represent a point with any particular velocity, with the constant V, which represents only one velocity– the velocity of light– we will obtain Einstein's equation $\xi = \varphi(v)\beta(x - vt)$. To accomplish this transformation we need to keep in mind the following particulars: a light beam travels with velocity $V - v$ relative to the origin of K, a represents the function $\varphi(v)$, the term x'/ω_ξ is equal to t, and that in our special case instant where $V = \omega_\xi$ the term t equals $x'/(V - v)$. Thus, we rewrite the equation $\xi = a\omega_\xi[(x'/\omega_\xi - v/(V^2 - v^2) \bullet x']$ as $\xi = aV[(x'/(V - v) - v/(V^2 - v^2) \bullet x']$. We follow the same pattern of mathematical manipulation we followed to generate Eq. (3.12) including incorrectly taking the square root of only one side of the equation. Following this procedure will produce Einstein's equation $\xi = \varphi(v)\beta(x - vt)$.

The equation $\xi = V\tau$ is a special case of the equation $\xi = \omega\tau$ where the term ω represents a point with any particular velocity. If we prefer to adhere strictly to Einstein's thought experiments, we would say $\xi = V\tau$ is a special case of $\xi = \omega_\xi\tau$ and $\eta = V\tau$ is a special case of $\eta = \omega_\eta\tau$.

If we examine a familiar example of a special case that occurs in arithmetic, we can appreciate the limitations of a special case. In division there occurs the special case of dividing by ten or a power of ten. The divisor can be 10, 100, 1000, . . . 10^n. The dividend can be any number. To divide any dividend by ten or a power of ten, express the divisor in the form 10^n and then move the decimal point of the of dividend n place values to the left to obtain the quotient. If the divisor is 10^{-n}, the decimal point of the dividend is shifted n place values to the right. For example, 6,321 ÷ 100 = 63.21; the decimal point of the dividend is moved two place values to the left to obtain the quotient. In this special case, the procedure of long division is greatly simplified, but this procedure is only valid with a divisor of ten or a power of ten. For example, we cannot obtain the quotient of 6,321 ÷ 127 by any kind of shifting of the decimal point.

Divisors of ten and divisors of a power of ten have something in common with all dividends. The place values that are used in the expression of the dividends are the following: 10, 100, 1000, . . . 10^n. In a similar fashion the term V in the equation $\xi = V\tau$ has something in common with the term τ when it is expressed with the quantities t and x' since $\tau = a[t - v/(V^2 - v^2) \cdot x']$, and also the term V has something in common with the term t when it is expressed as $t = x'/(V - v)$.

It is also interesting to note the way the velocity ω_ξ from the equation $\xi = \omega_\xi\tau$ behaves in Eq. (5.4), $x = (\omega_\xi + v)/[1 + (v\omega_\xi / V^2)] \cdot t$. The term ξ is transformed into $\beta(x - vt)$, and τ is transformed into $\beta(t - vx/V^2)$ to produce Eq. (5.4), but ω_ξ remains unchanged. The way the equation is presented gives the impression that ω_ξ has been transformed but it has not. It is packaged with the terms that are used to transform the variables ξ and τ into the variables x and t, and they provide the appearance of transforming ω_ξ. Viewed from the rest system K, the velocity ω_ξ would be augmented by the velocity v so that $\omega_{x\ (rest\ system)} = \omega_\xi + v$.

If we think of the moving system k as a long train of flat-bed cars pulled by a locomotive, we can picture ω_ξ as a runner running along the flat-bed cars. If we imagine the train was traveling slowly due east at 2 mph, and the runner running due west at 2 mph, to an observer standing on the railroad embankment, the runner would appear to be running in place. The observer standing on the railroad embankment represents an observer in the rest system K.

This situation is represented by the formula $\omega_\xi + v = 0$ or −2 mph due west + 2mph due east = 0 mph. When $\omega_\xi + v = 0$ the value for x is zero because $(0)/[1 + (v\omega_\xi/ V^2)] \cdot t = x = 0$. If we imagine that there is a runner on every flat-bed car running due west at 2 mph, according to Eq. (5.4) the x coordinate for each runner is zero, but this is clearly impossible. If the runner on the last flat-bed car was assigned an x coordinate of zero, that value could not be assigned to the others. Eq. (5.4), $x = (\omega_\xi + v)/[1 + (v\omega_\xi/ V^2)] \cdot t$, is incomplete. When the velocities ω_ξ and v cancel out Eq. (5.4) assigns all points the x coordinate of zero regardless of their true position on the X-axis. This is not surprising considering that Eq. (5.4) is constructed from equations that represent special cases. Eq. (5.4) needs the addition of a term such as x_0 to indicate the position of a point or a runner before they are set into motion.

We can now return to Einstein's discussion of a point in motion in the moving system k.

> Thus, according to our theory, the vector addition for velocities holds only to first approximation. Let
>
> $$U^2 = (dx/dt)^2 + (dy/dx)^2, \qquad (5.7)$$
>
> $$\omega^2 = \omega_\xi^2 + \omega_\eta^2 \qquad (5.8)$$
>
> and
>
> $$\alpha = \arctan \omega_\eta/\omega_\xi ; \qquad (5.9)$$
>
> α is then to be considered as the angle between the velocities v and ω. After a simple calculation we obtain
>
> $$U = \frac{[(v^2 + \omega^2 + 2v\omega \cos\alpha) - (v\omega \sin\alpha/ V)^2]^{1/2}}{1 + v\omega \cos\alpha/ V^2} . \qquad (5.10)$$

It is worth noting that v and ω enter into the expression for the resultant velocity in a symmetrical manner. If ω also has the direction of the X-axis (Ξ -axis), we get

$$U = \frac{v + \omega}{1 + v\omega/V^2}. \qquad (5.11)$$ [81]

It will be helpful to examine Eqs. (5.7) through (5.11) in detail. We will examine both the intermediate steps that are employed to produce the equations and also the validity of several of the equations.

To find dx/dt we treat all the terms on the right side of Eq. (5.4) as constants except for t. If we let $C = (\omega_\xi + v)/[1 + (v\omega_\xi/ V^2)]$, we can rewrite the equation as $x = Ct$. Therefore, $dx/dt = C = (\omega_\xi + v)/[1 + (v\omega_\xi/ V^2)]$ and $(dx/dt)^2 = (\omega_\xi^2 + 2\omega_\xi v + v^2)/[1 + (v\omega_\xi/ V^2)]^2$.

To find dy/dt we treat all the terms on the right side of Eq. (5.5) as constants except for t. If we let $C = [1 - (v^2/ V^2)]^{1/2} / [1 + (v\omega_\xi/ V^2)] \bullet \omega_\eta$, we can rewrite the equation as $y = Ct$. Therefore, $dy/dt = C$ or $dy/dt = [1 - (v^2/ V^2)]^{1/2} / [1 + (v\omega_\xi/ V^2)] \bullet \omega_\eta$, and $(dy/dt)^2 = [1 - (v^2/ V^2)] \bullet \omega_\eta^2/ [1 + (v\omega_\xi/ V^2)]^2 = [\omega_\eta^2 - \omega_\eta^2(v^2/ V^2)]/ [1 + (v\omega_\xi/ V^2)]^2$.

Since $(dx/dt)^2$ and $(dy/dt)^2$ have the same denominator, $[1 + (v\omega_\xi/ V^2)]^2$, we have, $U^2 =\{\omega_\xi^2 + \omega_\eta^2 + v^2 + 2v\omega_\xi - (v^2/ V^2)\omega_\eta^2\}/ [1 + (v\omega_\xi/ V^2)]^2$. Since $\omega^2 = \omega_\xi^2 + \omega_\eta^2$, we have $U^2 =\{\omega^2 + v^2 + 2v\omega_\xi - (v^2/ V^2)\omega_\eta^2\}/ [1 + (v\omega_\xi/V^2)]^2$. Eq. (5.8), $\omega^2 = \omega_\xi^2 + \omega_\eta^2$, represents the vector addition of the two components of vector ω. The component vectors are ω_ξ and ω_η. They are added together to produce a third vector, ω, which is called the resultant. The vector ω_ξ is the ξ-component of vector ω. The vector ω_η is the η-component of vector ω. The vectors ω_ξ and ω_η

are added together by a method known as the "Parallelogram of forces." It is also known as the "Parallelogram Law." Since vectors have both magnitude and direction, they cannot be added together in the same manner as scalars. For example, temperature is a scalar so 30°F. + 40°F. = 70°F. However, velocity is a vector so you cannot add the vectors 30 mph due east + 40 mph due north and obtain 70 mph due east and due north. The Parallelogram of forces is a method of adding vectors together that allows you to mesh or combine the magnitude and direction of one vector with the magnitude and direction of another vector. The resulting vector has a proportional mixture of the magnitude and direction of the two vectors that have been added together.

In Eq. (5.9), $\alpha = \arctan \omega_\eta/\omega_\xi$, the term arctan denotes an inverse trigonometric function, the $\arctan \omega_\eta/\omega_\xi$ is an angle α, whose tangent is ω_η/ω_ξ or $\tan\alpha = \omega_\eta/\omega_\xi$. Since $\tan\alpha = \sin\alpha/\cos\alpha$, it follows that $\sin\alpha/\cos\alpha = \omega_\eta/\omega_\xi$. Sinα in a given right triangle is the ratio of the length of the side opposite to angle α to the length of the hypotenuse, and therefore, $\sin\alpha = \omega_\eta/\omega$. Cosα in a given right triangle is the ratio of the length of the adjacent side of angleα to the length of the hypotenuse, and therefore, $\cos\alpha = \omega_\xi/\omega$.

We are now ready to generate Eq. (5.10). We have already established that $U^2 = \{\omega^2 + v^2 + 2v\omega_\xi - (v^2/V^2)\omega_\eta^2\}/[1 + (v\omega_\xi/V^2)]^2$.

The equation $\cos\alpha = \omega_\xi/\omega$ can be rewritten as $\omega_\xi = \omega\cos\alpha$, and therefore, the term $2v\omega_\xi$ becomes $2v\omega\cos\alpha$. Likewise the term $(v\omega_\xi/V^2)$ becomes $(v\omega\cos\alpha/V^2)$. The equation $\sin\alpha = \omega_\eta/\omega$ can be rewritten as $\omega_\eta = \omega\sin\alpha$, and therefore, the term $(v^2/V^2)\omega_\eta^2$ becomes $(v^2/V^2)(\omega\sin\alpha)^2$ or $(v\omega\sin\alpha/V)^2$. With these changes the equation becomes the following: $U^2 = \{\omega^2 + v^2 + 2v\omega\cos\alpha - (v\omega\sin\alpha / V)^2\} / [1 + v\omega\cos\alpha / V^2]^2$. If we take the square root of each side of the equation we obtain Eq. (5.10), which is as follows: $U = [\omega^2 + v^2 + 2v\omega\cos\alpha - (v\omega\sin\alpha / V)^2]^{½} / [1 + v\omega\cos\alpha / V^2]$.

To produce Eq. (5.11), $v + w/[1 + vw/ V^2]$, we assume that vector ω is traveling along the X-axis or as Einstein states, "If ω also has the direction of the X-axis (Ξ-axis) . . .."[82] When the vector ω travels along the X-axis, the angle α = 0°. Therefore, the cosα = 1 and the sinα = 0. When we insert those values into Eq. (5.10) we obtain the following: $U = [\omega^2 + v^2 + 2v\omega(1) - (v\omega(0) / V)^2]^{½} / [1 + v\omega(1)/ V^2]$. The simplified version of the equation is, $U = [\omega^2 + v^2 + 2v\omega]^{½} / [1 + v\omega/ V^2)]$. The square root of the term $[\omega^2 + v^2 + 2v\omega]$ is v + ω, and thus we have produced Eq. (5.11), $v + w/[1 + vw/ V^2]$.

This concludes our examination of the intermediate steps for Eqs. (5.7) through (5.11). Now, we will examine the validity of several of the equations.

The validity of Eq. (5.9), $\alpha = \arctan \omega_\eta/\omega_\xi$, can be questioned. Since ω is a vector it has both magnitude and direction. It follows that the component vector ω_η expresses both the magnitude and direction that vector ω has relative to the η-axis. It also follows that the component vector ω_ξ expresses both the magnitude and direction that vector ω has relative to the ξ-axis. For simplicity, let us denote that the direction along the η-axis that has increasingly positive y values is North, and also let us denote that the direction along the ξ-axis that has increasingly positive x' values is East. For example, ω_η could be a component vector with a velocity of 3 mph due North, and ω_ξ could be a component vector with a velocity of 4 mph due East. If we rewrite Eq. (5.9), α = arctan ω_η/ω_ξ, as tanα = ω_η/ω_ξ, we obtain tanα = 3 due North/4 due East. Since tangents are ratios with no units of measure attached to them, the value we want to obtain is tanα = 3/4. This same dilemma would be present for the values sinα = ω_η/ω and cosα = ω_ξ/ω. Since sinα = ω_η/ω, from our example sinα = 3 due North/5 Northeast by east. This dilemma may prove to be an insurmountable problem for the production of Eqs. (5.10) and (5.11).

Before we reach any conclusions about Eqs. (5.10) and (5.11), we must clarify the type of vector multiplication Einstein is employing in Eq. (5.9). Is Einstein employing the scalar multiplication of two vectors? The Mathematics Dictionary offers the following explanation of the scalar multiplication of two vectors:

> The scalar product of two vectors is the scalar which is the product of the lengths of the vectors and the cosine of the angle between them. This is frequently called the dot product, denoted by A • B, or the inner product. It is equal to the sum of the products of the corresponding components of the vectors.[83]

The definition above does away with the dilemma poised by the fact that vectors have direction by removing the directional component from the vector and thus producing a scalar. Unfortunately, it does introduce a complicating factor of its own by requiring that the product of the lengths of the vectors must be multiplied by the cosine of the angle between them. For example, since tanα = ω_η/ω_ξ, the dot product A • B would be as follows:[ω_η • 1/ω_ξ] • cos 90°, which results in tanα = 0. This is so because the angle between ω_η and ω_ξ is 90° and cos 90° = 0.

If Einstein were to use scalar multiplication of vectors for the multiplication of all his vectors, the result would be that all his vectors that were multiplied would lose their directional quality. It seems unlikely that Einstein's resultant velocity, U, could be obtained from vectors that have lost their directional component through scalar multiplication. The Mathematics Dictionary offers the following definition of a non-scalar method for multiplying vectors, and the method is denoted as the vector multiplication of two vectors. This method of multiplying vectors is also incompatible with Einstein's results:

> The vector product of two vectors A and B is the vector C whose length is the product of the lengths of A and B and the sine of the angle between them (The angle from the first to the second), and which is perpendicular to the plane of the given vectors and directed so that the three vectors in order A, B, C form a right-handed trihedral.[84]

Neither the scalar multiplication of two vectors nor the vector multiplication of two vectors produce results compatible with those required by Einstein. Einstein seems to treat the vectors in Eq. (5.9) as scalars. He disregards their directional component and employs the vectors exclusively as measures of length. He does this because it serves his purpose to do so. Since we have questioned the validity of Eq. (5.9), can we produce Eqs. (5.10) and (5.11) without the use of Eq. (5.9)?

As we have shown, without the use of Eq. (5.9) an only slightly different version of Eq. (5.10) can be obtained, i. e., $U^2 = \omega^2 + v^2 + 2v\omega_\xi - (v^2/V^2)\,\omega_\eta^2/[1 + (v\omega_\xi/V^2)]^2$. The only significant differences are ω_ξ instead $\omega\cos\alpha$ and ω_η instead of $\omega\sin\alpha$. To obtain Eq. (5.11) we set $\omega_\eta^2 = 0$. This serves two purposes, first it rids the above equation of the term $(v^2/V^2)\omega_\eta^2$, and secondly, when substituted in Eq. (5.8), $\omega^2 = \omega_\xi^2 + \omega_\eta^2$, we obtain $\omega = \omega_\xi$. These two changes allow us to produce Eq. (5.11) when we take the square root of each side of the equation.

But, do they really allow us to produce Eq. (5.11)? Can we set $\omega_\eta^2 = 0$ with no consequences? The answer is no. If we set $\omega_\eta^2 = 0$, then y in Eq. (5.5) equals zero and thus, $(dy/dx)^2$ equals zero. When $(dy/dx)^2$ equals zero Eq. (5.10) cannot be produced.

Since Eq. (5.10), $U = [\omega^2 + v^2 + 2v\omega\cos\alpha - (v\omega\sin\alpha/V)^2]^{1/2}/[1 + v\omega\cos\alpha/V^2]$, can only be produced using Eq. (5.9), we can question the validity of Eqs. (5.10) and (5.11). The flaw with Eq. (5.10) is that we replace two component vectors with their scalar counterparts. The component vector ω_ξ is replaced with its scalar counterpart $\omega\cos\alpha$, and the component vector ω_η is replaced with its scalar counterpart $\omega\sin\alpha$. As we have noted to produce Eq. (5.11), Einstein requires that $\alpha = 0°$. Thus, $\sin\alpha = 0$ and the term $(v\omega\sin\alpha/V)^2$ disappears. Since $\cos\alpha = 1$, the term $2v\omega\cos\alpha$ becomes merely $2v\omega$. What is the term $2v\omega$? It must be 2 multiplied by vector v multiplied by scalar ω. This is despite the fact that ω_ξ from the term $2v\omega_\xi$ is a vector. When we replace the term component vector ω_ξ with the term $\omega\cos\alpha$ from the equation $\omega_\xi = \omega\cos\alpha$, we are replacing a component vector with its scalar equivalent.

Thus, the interpretation of the equation $U = [\omega^2 + v^2 + 2v\omega]^{½} / [1 + v\omega\cos\alpha/ V^2]$ becomes problematic. The numerator of the right side of the equation should be interpreted in the following manner. The square root of the sum of the following three terms: vector ω^2 + vector v^2 + 2 multiplied by vector v multiplied by scalar ω. Since ω is a scalar, the square root of the three terms is not ω + v. Therefore, Eq. (5.11) is invalid because the numerator cannot be ω + v.

We could try to produce Eq. (5.11), $U = (v + \omega)/[1 + v\omega/V^2]$ by stating it is merely the velocity component of Eq. (5.4), $x = (\omega_\xi + v)/[1 + v\,\omega_\xi/ V^2] \bullet t$, where $\omega = \omega_\xi$. But, as we have pointed out before, Eq. (5.4) is not a valid equation. We could also try to maintain the validity of Eqs. (5.10) and (5.11) by claiming all the terms are scalars and not vectors. But, if we take away the directional component from the terms, it becomes impossible to calculate angle α, which is the angle between vector v and vector ω. We should note that both vector v and component vector ω_ξ run along the ξ-axis in the direction of increasing x'.

The preceding discussion may have seemed unduly confusing. There is an explanation for the confusion. In Thomas H. Barr’s textbook Vector Calculus he states that the division of vectors is not defined. He also states that under certain circumstances the multiplication of vectors is not defined. Since his examples of the correct determination of vector products and quotients use unit vectors, we must familiarize ourselves with that term. T. H. Barr provides a concise definition of unit vectors in “Chapter 1. Coordinate and Vector Geometry” of his textbook.

> Establish a rectangular coordinate system in three-dimensional space. Any vector a can be put into standard position by translating it so that its tail lies at the origin. We call the coordinates (a_1, a_2, a_3) of the head of a in standard position the components of a. We denote by i the vector with components (1, 0, 0), by j the vector with components (0, 1, 0) and k the vector with components (0, 0, 1). If a is any vector with components (a_1, a_2, a_3) then $a = a_1 i + a_2 j + a_3 k$.[85]

With the introduction of the unit vectors i, j, and k, we are ready to encounter T. H. Barr's examples, which demonstrate that vector division and certain types of vector multiplication are not defined. Since vector division is not defined, Eq. (5.9) is invalid. The type of vector multiplication that is not defined will be applicable to our analysis of Eq. (5.10), and the vector multiplication that is not defined, will invaidate Eq. (5.10). The following quotation is from question one of "Exercises 1.4," and only the answers relevant to our analysis are included.

> Let a = 5i − j, b = 7i − 3j + 2k, c = −4i + 9j − 8k,
> and d = −3i −11j + 7k.
> b/d not defined   ad not defined[86]

According to T. H. Barr, the division of vector b by vector d is not defined. Both vectors b and d are vectors defined in three dimensional space. We can extrapolate from T. H. Barr's example and conclude that vector division is not defined for all vectors regardless of the dimensionality of the space in which they occur. The term ω_η/ω_ξ states that component vector ω_η, which occurs in one dimension, is divided by component vector ω_ξ, which also occurs in one dimension. Thus, the term ω_η/ω_ξ is not defined. Therefore, Eq. (5.9), $\alpha = \arctan \omega_\eta/\omega_\xi$, is invalid.

Also, according to T. H. Barr, the multiplication of vector a by vector d is not defined. Vector a is a vector that occurs in two dimensions. Vector d is a vector that occurs in three dimensions. We can extrapolate from this example and conclude that the multiplication of a vector occurring in one dimension by a vector occurring in two dimensions is not defined, and thus, the term $2v\omega\cos\alpha$ from Eq. (5.10) is not defined. Therefore, Eq. (5.10) is invalid. The velocity vector v is a vector that occurs in one dimension. The velocity vector v is restricted to the X-axis of the system K. The velocity vector ω is vector that occurs in two dimensions. The velocity vector ω is restricted to the XY plane of the system K.

We can use T. H. Barr's examples of vector operations that are not defined to analyze an ambiguous statement made by Einstein regarding Eq. (5.10). Directly following Eq. (5.10) Einstein states, "It is worth noting that v and ω enter into the expression for the resultant velocity in a symmetrical manner."[87] The expression for the resultant velocity is Eq. (5.10), $U = [\omega^2 + v^2 + 2v\omega\cos\alpha - (v\omega\sin\alpha / V)^2]^{½} / [1 + v\omega\cos\alpha / V^2]$. According to the Mathematics Dictionary, "a symmetric relation is a relation which has the property that if a is related to b, then b is related in like manner to a. The equals relation of algebra is symmetric, since if a = b, then b = a."[88] Since vω = not defined and not defined = vω, we can say vω and not defined form a symmetric relation. The Mathematics Dictionary also states, “A relation is asymmetric if there are no pairs (a, b) such that a is related to b and b is related to a. The property of being older than is asymmetric; if a is older than b, then b is not older than a.”[89] Since vector ω occurs in one more dimension than vector v and vector v does not occur in one more dimension than vector ω, we can say the relation between vector ω and vector v is asymmetric with regard to their dimensionality. This analysis has not given a precise meaning to Einstein’s ambiguous statement, but it has shown there are several ways with which further interpretation could precede.

This concludes are analysis of the validity of Eqs. (5.7) through (5.11). We can now turn our attention to Einstein’s analysis of Eq. (5.11).

It is from an analysis of this invalid equation that Einstein determines two fundamental principles. The first principle is that adding together two velocities that are smaller than the velocity of light results in a velocity that is smaller than the velocity of light. The second is that the velocity of light cannot be changed by adding the velocity of light to any sub-light velocity.

We can continue with Einstein’s examination of Eq. (5.11), $(v + \omega)/[1 + v\omega/V^2]$ to show the method he used to demonstrate the two principles noted above.

> It follows from this equation that the composition of two velocities that are smaller than V always results in a velocity that is smaller than V. For if we set $v = V - \kappa$, and $\omega = V - \lambda$, where κ and λ are positive and smaller than V, then
>
> $$U = V \frac{2V - \kappa - \lambda}{2V - \kappa - \lambda + \kappa\lambda / V} < V. \qquad (5.12)$$ [90]

Eq. (5.12) states that U equals the velocity of light, V, multiplied by some fraction (such as .9 for instance) and thus, the resulting value is less than the velocity of light V.

The numerator of Eq. (5.12) comes from the addition of $v + \omega$, and since $v = (V - \kappa)$ and $\omega = (V - \lambda)$, we have $(V - \kappa) + (V - \lambda)$. This sum can be rewritten as $2V - \kappa - \lambda$.

The denominator of Eq. (5.12) is $1 + v\omega/V^2$, which can be rewritten as a single fraction with the common denominator V^2 and with the same substitutions as above for v and ω. The result is $[V^2 + (V - \kappa)(V - \lambda)]/ V^2$ or $[2V^2 - V\kappa - V\lambda + \kappa\lambda]/ V^2$. If we divide each term separately by V^2, the denominator can be rewritten in this form $2 - \kappa/V - \lambda/V + \kappa\lambda/V^2$. The equation now assumes the form of $U = [2V - \kappa - \lambda]/ [2 - \kappa/V - \lambda/V + \kappa\lambda/ V^2]$. If we multiply the right side of the equation by one in the form of V/ V, we obtain $U = V [2V - \kappa - \lambda]/ [2V - \kappa - \lambda + \kappa\lambda/V]$, which is less than V. This is so because $[2V - \kappa - \lambda]$ is a smaller number than $[2V - \kappa - \lambda + \kappa\lambda/V]$, and a smaller number divided by a larger number gives us a fraction that is less than one.

Einstein continues in his examination of Eq. (5.11).

> It also follows that the velocity of light V cannot be changed by compounding it with a "subluminal velocity." For this case we get
>
> $$U = \frac{V + \omega}{1 + \omega/V} = V \qquad (5.13)$$ [91]

Eq. (5.13) is obtained by letting $v = V$ in Eq. (5.11). The numerator becomes $V + \omega$. The denominator becomes $1 + V\omega/V^2$ or $1 + \omega/V$. The denominator can be written as a single fraction $(V + \omega)/V$. Thus the equation becomes $U = (V + \omega)/[(V + \omega)/V]$. If we multiply the right side of the equation by one in the form of V/V, we obtain $U = V(V + \omega)/(V + \omega)$. It is apparent that the right side of the equation equals V because the term V is multiplied by the same quantity, $(V + \omega)$, that it is divided by $(V + \omega)$. Thus we obtain $U = V(V + \omega)/(V + \omega) = V$. If we multiply the middle term of the previous equation, $V(V + \omega)/(V + \omega)$, by one in the form of $(1/V)/(1/V)$, we obtain Eq. (5.13), $U = (V + \omega)/[1 + \omega/V] = V$. In Eq. (5.12) a fraction is multiplied by one in the form of V/V and in Eq. (5.13), as we have just seen, a fraction is multiplied by one in the form of $(1/V)/(1/V)$.

The final point Einstein makes in the fifth section involves the introduction of another coordinate system k'. Since the fifth section is the last section in "A. Kinematic Part," it is the final point we will discuss. As we have noted K is the rest system. The moving system k moves with velocity v relative to the rest system K in the direction of increasing x of the rest system K. The X-axes of the two systems coincide and the X-axis of the k system is denoted as the ξ-axis. The third coordinate system k' moves with velocity ω relative to the moving system k in the direction of increasing ξ. The X-axis of the k' system coincides with the X-axes of the rest system K and the moving system k; and is denoted as the Ξ-axis. All three axes coincide the X, ξ and Ξ. The velocity of the k' system relative to the resting system K is as follows: $(v + \omega)/[1 + v\omega/V^2]$.

In this scenario, three axes coincide, and each axis is from a different system. This is difficult to imagine since Einstein describes each system as consisting of, "three mutually perpendicular rigid material lines originating from one point."[92] The three rigid material lines that form the X, ξ, and Ξ axes cannot coincide.

Part Five

Conclusion: A Summary of the Mathematical Inconsistencies in Part A. the "Kinematic Part" of On the Electrodynamics of Moving Bodies

The Eq. (1.1), $t_B - t_A = t'_A - t_B$ is invalid for moving systems, if we adopt the conventional definition of distant synchronized clocks. The conventional definition requires that the clocks agree in the time that they keep. In a moving system, when a beam of light makes a round-trip journey from point A to a mirror at point B, where it is reflected back to point A, the duration of the outbound journey, represented by $t_B - t_A$, is not equal to the duration of the return journey, represented by $t'_A - t_B$.

The Eq. (3.1), $\frac{1}{2}\{\tau[0, 0, 0, t] + \tau[0, 0, 0, t + x'/V - v) + x'/V + v)]\} = \tau[x', 0, 0, t + x'/V - v)]$, is a complicated restatement of Eq. (1.1), and thus, it is also invalid for moving systems, if we adopt the conventional definition of distant synchronized clocks. The equation states that the total duration of a light beam's round-trip journey multiplied by ½ is equal to the duration of the outbound portion of the light beam's journey.

The Eq. (3.2), $\frac{1}{2}[1/(V - v) + 1/(V + v)]\partial\tau/\partial t = \partial\tau/\partial x' + [1/(V - v)]\,\partial\tau/\partial t$, is invalid because it cannot be legitimately derived from Eq. (3.1). Since Eq. (3.3), $\partial\tau/\partial x' + v/(V^2 + v^2)\,\partial\tau/\partial t = 0$ is merely a rearranged version of Eq.(3.2), it also is invalid because it cannot be legitimately derived from Eq.(3.1).

The unnumbered equation, $\partial\tau/\partial y = 0$, and the unnumbered equation, $\partial\tau/\partial z = 0$, are both invalid because they cannot be legitimately derived from Einstein's two implicitly reformulated versions of Eq. (3.1). The first reformulated version of Eq. (3.1) fails to produce equation $\partial\tau/\partial y = 0$, and the second reformulated version of Eq. (3.1) fails to produce equation $\partial\tau/\partial z = 0$.

The Eq. (3.4), $\tau = a[t - v/(V^2 + v^2) \cdot x']$, is invalid because Einstein incorrectly claims that some fashion of integration, when applied to the three partial derivative equations listed above, yields this equation. The three partial derivative equations: $\partial\tau/\partial x' + v/(V^2 + v^2)\,\partial\tau/\partial t = 0$, $\partial\tau/\partial y = 0$, and $\partial\tau/\partial z = 0$ do not generate Eq. (3.4) through some fashion of integration.

The four transformation equations are invalid. Einstein incorrectly claims all four of these equations can be produced through combining Eq. (3.4) with various simple equations. Each of these simple equations describes the distance a beam of light travels. The beam of light makes a separate journey along each one of the three axes of the moving system k. The four transformation equations are the following: Eq. (3.11) $\tau = \varphi(v)\beta(t - vx/V^2)$, Eq. (3.12) $\xi = \varphi(v)\beta(x - vt)$, Eq. (3.13) $\eta = \varphi(v)y$, and Eq. (3.14) $\zeta = \varphi(v)z$.

Einstein does not demonstrate his claim that Eq. (3.16), $x^2 + y^2 + z^2 = V^2t^2$, can be transformed, using the transformation equations, into Eq. (3.17), $\xi^2 + \eta^2 + \zeta^2 = V^2\tau^2$. Einstein does not provide us with the details of this calculation in On the Electrodynamics of Moving Bodies. In "Appendix One" of his book Relativity: the Special and the General Theory he does provide a version of this calculation, but the calculation is invalid.

The unnumbered term from the paper's fourth section $\frac{1}{2} t(v^2 / V^2)$ sec., which describes the amount of time a clock transported from A to B will lag behind a clock located at point B where t is the time needed by the clock to travel from A to B, is invalid for three reasons. The first reason is a minor point. There is no need for a unit of time measured in seconds to be appended to the term since the variable t is a unit of time measured in seconds. The second and third reasons also apply to a similar unnumbered term, which is also from the paper's fourth section $\frac{1}{2} (v^2 / V^2)$ sec. The second reason is that both of these terms are approximations of the term $[1 - (1 - v^2/ V^2)^{\frac{1}{2}}]$ sec., but Einstein does not acknowledge the terms are approximations. He seems to ambiguously imply they are not approximations. The third reason is that both terms do not take into account Einstein's definition of distant synchronized clocks. According to Einstein's definition, distant synchronized clocks do not need to agree in the time they keep and further will not agree in the time they keep in a moving system. Yet, these two terms operate on the assumption that distant synchronized clocks in a moving system agree in the time they keep.

The Eq. (5.4), $x = (\omega_\xi + v)/[1 + (v\omega_\xi / V^2)] \cdot t$, is incomplete. The equation assigns every instance of the occurrence of $\omega_\xi + v = 0$ to a value of $x = 0$ regardless of the occurrence's actual position on the X-axis. The Eq. (5.4) is also invalid because it is composed of many special case instances that do not apply to a point moving with a constant velocity in the moving system k. For example, Eq. (5.4) is partially composed from an equation that describes a beam of light traveling in a specific direction in the moving system k, as viewed from the rest system K. In this specific instance, because of the properties of light beams and other conditions, the velocity of the light beam when viewed from the rest system K is $V - v$. However, a material point, traveling under the same conditions, with the constant component of velocity of ω_ξ, would have a constant, component velocity of $\omega_\xi + v$, when viewed from the rest system K. This is because a material point is carried along by the moving system k and a light beam is not carried along by the moving system k.

The Eq. (5.9), $\alpha = \arctan \omega_\eta/\omega_\xi$ is invalid because component vectors have direction as well as magnitude. The magnitude of ω_η/ω_ξ is equal to $\tan\alpha$, but there is no method of vector multiplication that can get rid of the directional component of the vector without causing further complications.

The Eq. (5.10), $U = [\omega^2 + v^2 + 2v\omega\cos\alpha - (v\omega\sin\alpha / V)^2]^{1/2} / [1 + v\omega\cos\alpha / V^2]$, is invalid because component vectors ω_η and ω_ξ are replaced by $\omega\sin\alpha$ and $\omega\cos\alpha$, respectively. Both of which are produced by the scalar multiplication of two vectors, which results in a scalar quantity, and hence, these two terms have no directional component.

The Eq. (5.11), $U = (v + \omega)/[1 + v\omega/V^2]$, is not valid because $(v + \omega)$ is not the square root of $\omega^2 + v^2 + 2v\omega$, although at first glance it may appear to be. This is the case for many complicated reasons. One reason is that $(v + \omega)^2$ gives us the following: (vector v^2) + (vector ω^2) + (vector v • vector ω) + (vector ω • vector v). The term $(v + \omega)^2$ is not equal to $\omega^2 + v^2 + 2v\omega$, which gives us the following: (vector v^2) + (vector ω^2) + 2 (vector v • scalar ω).

The concluding assessment of the "Kinematic Part" of On the Electrodynamics of Moving Bodies is twofold in nature. First, the thought experiments are ambiguous and inconclusive as a partial result of definitions that are ambiguous and contradictory. Secondly, the majority of the mathematical equations introduced by Einstein are invalid.

Note on Eqs. (3.18) and (3.19)

A typographic error occurs in Einstein's Eqs. (3.18) and (3.19). Previously, this typographic error has not been formally acknowledged. In the central portions of both Eqs. (3.18) and (3.19) there is an erroneous duplication of the term (−v). Einstein presents the equations as the following: "$t' = \varphi(-v)\beta$

$(-v)[\tau + v/V^2 \cdot \xi] = \varphi(v)\varphi(-v)t$, (3.18) and $x' = \varphi(-v)\beta(-v)[\xi + v\tau] = \varphi(v)\varphi(-v)x$, (3.19)."[93] In the text of his book Albert Einstein's Special Theory of Relativity, A. I. Miller presents the equations as the following: "$t' = \varphi(-v)\gamma[\tau + v/V^2 \cdot \xi] = \varphi(v)\varphi(-v)t$, (6.35) and $x' = \varphi(-v)\gamma[\xi + v\tau] = \varphi(v)\varphi(-v)x$, (6.36)."[94] The term β is replaced with the term γ. Calculating the value of γ reveal that it is equal to β, which equals $1/(1 - v^2/V^2)^{1/2}$. The point to note is that there is no $(-v)$ following the term γ in both Eqs. (6.35) and (6.36).

The appendix of A. I. Miller's book, which contains his translation of On the Electrodynamics of Moving Bodies, reproduces Einstein's erroneous versions of the equations: "$t' = \varphi(-v)\beta(-v)[\tau + v/V^2 \cdot \xi] = \varphi(v)\varphi(-v)t$, (3.18) and $x' = \varphi(-v)\beta(-v)[\xi + v\tau] = \varphi(v)\varphi(-v)x$, (3.19)."[95] There is no acknowledgment of the typographic error by the use of a footnote or any other method.

Endnotes for the Introduction,
Parts One, Two, Three, Four, Five,
and the Note on Eqs. (3.18) and (3.19)

1. Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911) (London: Addison-Wesley Publishing Company, Inc., 1981), p.2.

2.Ibid., p.4.

3.Ibid., p. 123.

4.Ibid., p. 124.

5.Ibid., p. 7.

6.Ibid., p. 124.

7.Albert Einstein, Relativity: The Special and the General Theory (New York: Three Rivers Press, 1961), pp. 26–27.

8.Ibid., p. 27.

9.Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911) (London: Addison-Wesley Publishing Company, Inc., 1981), p. 7.

10.Christopher Jon Bjerknes, Anticipations of Einstein: In the General Theory of Relativity (Downers Grove, Illinois:XTX Inc., 2003), p. 7.

11.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p.101.

12.Ibid., p. 129.

13.Albert Einstein, Relativity: The Special and the General Theory (New York: Three Rivers Press, 1961), p.31.

14.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 101.

15.Ibid., p. 102.

16.Ibid., p. 124.

17.Ibid., p. 124.

18.Ibid., p. 125.

19.Ibid., p. 126.

20.Ibid., pp. 123–124.

21.Ibid., p. 125.

22.Ibid., p. 125.

23.Ibid., p. 124.

24.Ibid., p. 125.

25.Ibid., p. 126.

26.Ibid., p. 127.

27.Ibid., p. 127.

28.Ibid., p. 126.

29.Ibid., p. 127.

30.Ibid., p. 128.

31.Ibid., p. 129.

32.Ibid., pp. 126–127.

33.Ibid., pp. 129–130

34.Ibid., p. 127.

35.Ibid., p. 127.

36.Ibid., p. 127.

37.Ibid., pp. 129–130.

38.Ibid., p. 128.

39.Ibid., p. 132.

40.Ibid., p. xv.

41.Ibid., p. 132.

42.Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1909) (London: Addison-Wesley Publishing Company, Inc.,1981), p. 209.

43.Larry J. Goldstein and others, Calculus and Its Applications 4th ed. (Englewood Cliffs: Prentice-Hall Inc., 1987), p. 351.

44.Ibid., p. 378.

45.Ibid., p. 378.

46.Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1909) (London: Addison-Wesley Publishing Company, Inc.,1981), p. 209.

47.Glenn James and Robert C. James, Mathematics Dictionary, 3rd edition (Princeton: D. Van Nostrand Company, Inc., 1959), p. 273.

48.Ibid., p. 113.

49.Ibid., p. 389.

50.Ibid., p. 389.

51.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 132.

52.Ibid., p. 132.

53.Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911) (London: Addison-Wesley Publishing Company, Inc., 1981), p. 210.

54.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 133.

55.Larry J. Goldstein and others, Calculus and Its Applications 4th ed. (Englewood Cliffs: Prentice-Hall Inc., 1987), p. 340.

56.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 133.

57.Ibid., pp. 133–134.

58.Arthur I. Miller, Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911) (London: Addison-Wesley Publishing Company, Inc., 1981), p. 212.

59.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), pp. 134–135.

60.Albert Einstein, Relativity: The Special and the General Theory (New York: Three Rivers Press, 1961), pp. 131–132.

61.Ibid., p. 131.

62.Ibid., p. 134.

63.Lillian R. Lieber, The Einstein Theory of Relativity (New York: Rinehart & Company, Inc., 1936), p. 62.

64.Ibid., pp. 62–63.

65.Ibid., p. 64.

66.Ibid., pp. 64–65.

67.Ibid., p. 65.

68.Ibid., p. 66.

69.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 135.

70.Ibid., p. 136.

71.Ibid., p. 136.

72.Ibid., p. 137.

73.Ibid., p. 137.

74.Ibid., p. 138.

75.Ibid., pp. 137–138.

76.Arthur I. Miller, Albert Einstein's Special theory of Relativity: Emergence (1905) and Early Interpretation (1905–1909) (London:Addison-Wesley Publishing Company, Inc., 1981), p. 402.

77.Ibid., p. 402.

78.Ibid., p. 223.

79.John Stachel, Einstein's Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 140.

80.Ibid., p. 133.

81.Ibid., p. 141.

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83.Glen James and Robert C. James, Mathematics Dictionary 3rd edition (Princeton: D. Van Nostrand Company, Inc., 1959), p. 263.

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87.John Stachel, Einstein’s Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 141.

88.Glenn James and Robert C. James, Mathematics Dictionary, 5th edition (New York: Chapman & Hill, 1992), p. 411.

89.Ibid., p. 411.

90.John Stachel, Einstein’s Miraculous Year: Five Papers That Changed the Face of Physics (Princeton: Princeton University Press, 1998), p. 141.

91.Ibid., p. 141.

92.Ibid., p. 130.

93.Ibid., p. 135.

94.Arthur I. Miller, Albert Einstein’s Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1909) (London: Addison-Wesley Publishing Company, Inc., 1981), pp. 212–213.

95.Ibid., p. 400.