After studying the appropriate portions of Gödel, Escher, Bach: An Eternal Golden Braid, I have reached the conclusion that the author's depiction of Gödel's theorem contains a very significant error. It is the same error I detected in Gödel's Proof by James R. Newman and Ernest Nagel. If this same error is present in Kurt Gödel's original paper, On Formally Undecidable Propositions of Principia Mathematica and Related Systems I, Gödel's theorem is incorrect. Douglas R. Hofstadter's book Gödel, Escher, Bach is a monumental work that is 777 pages in length. It was originally published by Basic Books in 1979, but a Vintage Books (a division of Random House, Inc.) edition was published in 1989. His book presents a uniquely personal interpretation and explanation of Gödel's theorem among many other things.

In chapter 14 of his book, which is entitled "On Formally Undecidable Propositions of TNT and Related Systems 1," D. Hofstadter gives a concise derivation and explanation of Gödel's theorem. Hofstadter uses many terms that are unfamiliar, and some of these terms are portmanteau words of his own coinage.

His error, which occurs on page 447, involves the use of a portmanteau word of his own coinage. The incorrect statement is: a′ is the arithmoquinification of u. The correct statement is: a′ is the arithmoquinification of the numeric value of the Gödel number u, which is the Gödel number of a specific formula.

On page 446 Hofstadter writes, “ARITHMOQUINE {a′′, a′}   What the above formula says, in English, is: a′ is the Gödel number of the formula gotten by arithmoquining the formula with Gödel number a′′.” This definition requires that for arithmoquining to occur a′′ must be a formula with a Gödel number. On page 445 Hofstadter introduces another requirement for arithmoquining to occur: the formula must have at least one free variable. He writes, “Let us see an example of arithmoquining. We need a formula with at least one free variable. The following will do: a = S0. This formula’s Gödel number is 262,111,123,666, and we will stick this number into the formula itself--or rather, we will stick its numeral in. Here is the result: SSSSS.....SSSSS0 = S0 where there are 262,111,123,666 S’s on the left side of the equation. This new formula asserts a silly falsity--that 262,111,123,666 equals 1.”

It should be noted that Hofstadter has his own mechanism for determining the Gödel number of an equation. And, more importantly, it should be noted that for arithmoquining to occur a′′ must represent the

Gödel number of a formula with at least one free variable.

Also it should be noted that numerals are represented by combining the terms 0 (zero) and S (the immediate successor of) so that 1 = S0 and 2 = SS0 and 3 = SSS0 and so on. If Hofstadter wanted to replace the a′′ in the formula ARITHMOQUINE {a′′, a′} with the numeral 3 he would write: ARITHMOQUINE {SSS0/a′′, a′}. I cannot find a quotation that expresses that precise sentiment, but on page 442 he writes, "... to indicate the replacement of every a′ by a numeral, we write that numeral followed by "/a′"): ∃a:MIU-PROOF-PAIR {a,SSSSSSSSSSSSSSSSSSSSSSSSSSSSSS0/a′} Count the S's: there are 30." It is clear that in replacing a′′ with a numeral we would follow the same procedure that is used for replacing a′ with a numeral.

There are a few more terms that need to be mentioned before I can introduce Hofstadter's version of Gödel's theorem. TNT is an abbreviation for Typographic Number Theory. TNT-PROOF-PAIR {a, a′} refers to two natural numbers, a and a′ respectively. They form a TNT-PROOF-PAIR if and only if a is the Gödel number of a TNT-derivation whose bottom line is the string with Gödel number a′.

Gödel’s theorem is derived by arithmoquining a formula that Hofstadter refers to as the “uncle” formula. As a preface to Gödel’s theorem he writes this on page 447, “Now all we need to do is--arithmoquine this very uncle! What this entails is ‘booting out’ all the free variables--of which there is only one, namely a′′--and putting in the numeral for u everywhere. This gives us:

~∃a:∃a′:< TNT-PROOF-PAIR {a, a′}∧ARITHMOQUINE {SS...SS0/a′′,a′}>

where the number of S’s equals the numeral for u.” That is Hofstadter’s version of Gödel’s theorem or G. On page 447 he offers this interpretation of the theorem, “There do not exist numbers a and a′ such that both (1) they form a TNT-proof-pair, and (2) a′ is the arithmoquinification of u.” But, as I have pointed out Gödel’s theorem does not declare part (2) of his interpretation. Instead, the correct interpretation of part (2) is: a′ is the arithmoquinification of the numeral of the Gödel number u. The numeral of the Gödel number u cannot be arithmoquined because it is not a formula and therefore has neither a Gödel number nor a free variable. It seems to be incontrovertible that the numeral or numeric value (of the Gödel number u), which is expressed as {SS...SS0} where the number of S’s equals numeric value of Gödel number u, cannot be arithmoquined.

A corrected interpretation of Gödel's theorem as presented by Hofstadter might read: There do not exist numbers a and a′ such that both (1) they form a TNT-proof-pair, and (2) a′ is the arithmoquinification of a numeral (numeric value) that cannot be arithmoquined. Therefore, the number a′ does not exist. Or more simply, if a number a′ doesn't exist, the numbers a and a′ cannot be grouped together to form a pair of numbers.

There are a few remaining important points to cover. These will explain why Hofstadter's incorrect statement is so important to his argument and why Gödel's conclusion is so significant. The Gödel number of Gödel's theorem as presented by Hofstadter is: the arithmoquinification of u. That is why it is so important to Hofstadter that a′ is also the arithmoquinification of u. Thus both the formula (G) and an element of that same formula (a′) have the same Gödel number. This is a violation of the definition of Gödel numbers because every series of formulas and every formula and all the mathematical elements that are used to construct formulas have their own unique Gödel numbers. Perhaps, someone would argue a′ has a unique Gödel number as well as another Gödel number that it shares with Gödel's theorem (G).

When both a′ and Gödel's theorem (G) have the same Gödel number any statement about a′ is also statement about (G). So using a strategy of beginning with statements about a′ and then switching to statements about (G) Hofstadter accomplishes his goal of proving the incompleteness of TNT. He writes on pages 447 and 448, “There is no number a that forms a TNT-proof-pair with the arithmoquinification of u....The formula whose Gödel number is the arithmoquinification of u is not a theorem of TNT....G is not a theorem of TNT....But G’s nontheoremhood is what G asserts--hence G asserts a truth. And since G is not a theorem, there exists (at least) one truth which is not a theorem of TNT.” And, what Hofstadter demonstrates for TNT (Typographic Number Theory), Gödel theorem demonstrates for all but the most trivial formal systems for arithmetic of mathematical interest.

To make it explicit that a′ is not the arithmoquinification u, and that (G) is the arithmoquinification of u , we must examine the Gödel number u and the formula it represents. Here is the formula it represents:

~∃a:∃a′:<TNT-PROOF-PAIR {a, a′′}∧ ARITHMOQUINE {a′′,a′}>

Here is a portion of Gödel number u provided by Hofstadter on page 447, “u = 223,333,262,636, 333,262, 163,636,212,....161,....,213 For the rest we’d have to know just how the formulas TNT-PROOF-PAIR and ARITHMOQUINE actually look when written out.”

If we substitute the numeric value of Gödel number u for a′′ in the above formula, we produce Hofstadter's version of Gödel's theorem. Because the terminology is confusing, we should remind ourselves that the numeric value of the Gödel number u is: (SS...SS0) where the number of S's equal the numeric value of u. Thus, Gödel's theorem is the arithmoquinification of u.

On the other hand, the statement: a′ is the arithmoquinification of u is represented thus: ARITHMOQUINE {223,333, 262,636,333, 262,163, 636, 212,....161,....213/a′′,a′}. It is definitely not represented thus:

ARITHMOQUINE {SS...SS0/a′′,a′}.

It may be tempting to think that the problem could be resolved by exchanging the Gödel number u for the numeric value of the Gödel number u in Gödel's theorem. This would indeed rectify Hofstadter's incorrect statement: a′ is the arithmoquinification of u, but it would also require that the Gödel number of Gödel's theorem must no longer be the arithmoquinification of u. And, that change would serve to undermine the meaning of Gödel's theorem, as well.

It is difficult to say how much significance should be attached to the fact that D. Hofstadter and J. Newman along with his co-author E. Nagel make the same mistake. Another interesting feature is that Gödel numbers were heralded as a fundamental mathematical insight, yet apparently they have not lead to any other significant breakthroughs in 70 years since Gödel introduced them.