# A Critique of Kurt Gödel’s 1931 Paper: *On formally undecidable propositions of Principia Mathematica and related systems I*

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# Introduction

The translation we are using for the critique is by Martin Hirzel. It is available at: http://www.research.ibm.com/people/h/hirzel/ papers/canon00-goedel.pdf. The translation is dated November 27, 2000. The paper is 22 pages in length, and Hirzel states, "This document is a translation of a large part of Gödel's proof . . . This translation omits all footnotes from the original, and only contains sections 1 and 2 (out of four)."[1] The omitted sections are entitled "Generalizations" and "Implications for the nature of consistency." The two translated sections are entitled: "1–Introduction," which is two pages in length and "2–Main Result," which is 15 pages in length. The "Main Result" is divided into seven subsections: "Definitions," "Gödel-numbers," "Primitive recursion," "Expressing metamathematical concepts," "Denotability and provability," "Undecidibility theorem" and "Discussion."

I have concluded that the paper is a shell game. It is based on several errors that are well camouflaged. Some shortcomings in the paper are openly admitted although they are downplayed, and errors are also produced in an effort force a particular conclusion. This critique is limited to Gödel's first incompleteness theorem since that is the point at which Martin Hirzel's translation ends.

Torkel Franzén, in his book Gödel's Theorem: An Incomplete Guide to Its

Use and Abuse, defines Gödel's first incompleteness theorem. On page 16 of his book, he states, "Any consistent formal system S within which a certain amount of elementary arithmetic can be carried out is incomplete with regard to statements of elementary arithmetic: there are such statements which can neither be proved, nor disproved in S."[2]

## Part I

Gödel's paper is the description of a formal system he denotes with the letter P. Among other things it includes the natural numbers and the number zero. The first error occurs in subsection "2.1–Definitions." Gödel introduces the term "sign of type n" where n can stand for the individual natural numbers 1, 2, 3, etc. The category, sign of type n, includes both number-signs and variables. Number-signs are Gödel's term for the natural numbers including zero, which are considered constants in standard mathematics. In Gödel's system, constants have an almost entirely different definition.

It will be helpful to re-familiarize ourselves with some standard mathematical terms. A constant is either an integer or a variable whose domain is strictly limited. The domain of a variable that is a constant must be a set with only one member. An example of a domain with only one member is the following: States of the United States whose name begins with the letter

"D." Integers include the positive and negative whole numbers and zero. The domain of a variable is different from its solution set. The solution set is all the values of the variable that make a particular formula true. The solution set to a particular formula may often have only one member. A variable's domain includes all the possible solutions to a formula regardless of whether they are true or false. The distinction between a constant and a variable is well defined in standard mathematics.

In Gödel's system P the distinction is ambiguous. There are two possibilities for a sign of type n when 1 is substituted for n. On page 4 Gödel states, "In the first case we call such a sign a number-sign."[3] The designation for the "second case" is not made explicit. A sign of type 1 includes number-signs, which we would interpret in standard mathematics as the natural numbers plus zero, i.e., constants. In addition, a sign of type 1 includes <u>something</u> <u>else</u> that is not explicitly defined, but it may be a variable of type 1. In subsection "2.1–Definitions" Gödel ambiguously delineates the definitions of sign of type 1 (which includes number-signs and something else) and variable of type 1. The variables of type 1 in Gödel's system are not variables whose domains are sets with only one member. If they were variables whose domains are sets with only one member, that alone would spell the end of his proof. Terms that would be considered in standard mathematics constants and variables in the usual sense, i.e., variables whose domain is larger than a set with one member, are ambiguously included in the same grouping, which is called a sign of type 1 in Gödel's system P. This ambiguity is codified by definition. The ambiguous relationship between a sign of type 1 and a variable of type 1 is central to Gödel's proof.

An excerpt from "2.1–Definitions" of several paragraphs in length, translated from the original paper by M. Hirzel, should demonstrate the ambiguity of the situation.

"The basic signs of system P are the following:

I. Constant: "$\sim$" (not), "$\vee$" (or), "$\forall$" (for all), "0" (zero), "succ" (the successor of), "(" , ")" (parentheses).

II. Variable of type one (for individuals, i.e., natural numbers including 0): "$x_1$", "$y_1$", "$z_1$" . . .

Variables of type two (for classes of individuals): "$x_2$", "$y_2$", "$z_2$" , . . .

Variables of type three (for classes of classes of individuals) "$x_3$", "$y_3$", "$z_3$" , . . .

And so on for every natural number as type. . . .

By a sign of type 1 we understand a combination of signs of the form: a, succ(a), succ(succ(a)), succ(succ(succ(a))), . . . etc., where a is either 0 or a variable of type 1. In the first case we call such a sign a number-sign. For n > 1 we will understand by a sign of type n a variable of type n."[4]

And there is the ambiguity. Gödel never explains what he will call a sign of type 1 that is a member of the "second case" where a is a variable of type 1. He skips over the "second case" and moves on to define a sign of type n where n > 1. Gödel never explicitly states that the category a sign of type 1 includes both number-signs (constants) and variables of type 1. Also, he never explicitly defines whether natural numbers including zero refers to the domain or solution set of variables of type 1. It will be helpful to continue with this passage from Gödel's paper because he will shortly introduce the term generalization which will in turn imply the term substitution. Gödel will remind us repeatedly that when substituting b for the variable v it is crucial that b must be a sign of the same type as v. This ignores the fact that he hasn't made it clear if a sign of type 1 includes both number-signs and variables of type 1, i.e., both constants and variables (with domains larger than a set with one member).

Gödel continues, "We call combinations of signs of the form a(b), where b is a sign of type n and a is a sign of type n + 1, elementary formulae. We define the class of formulae as the smallest set that contains all elementary formulae and that contains for a and b always $\sim$(a), (a) $\vee$ (b), and $\forall x \cdot$ (a) (where x is an arbitrary variable). We call (a) $\vee$ (b) the disjunction of a and b, $\sim$(a) the negation and $\forall x \cdot$ (a) the generalization of a. A formula that contains no free variables (where free variables is interpreted in the usual manner) is called proposition-formula. We call a formula with exactly n free individual-variables (and no other free variables) a n-ary relation sign, for n = 1 also the formula is called a class-sign.

"By subst a $({}_{b}^{v})$ (where a is a formula, v is a variable and b is a sign of the same type as v ) we understand the formula that you get by substituting b for every free occurrence of v in a."[5]

Subst a $({}_{b}^{v})$ seems to be entirely innocuous, a mere bookkeeping formality, which states that one variable may be substituted for another variable as long as they are signs of the same type. In standard mathematics, variables can be classified by the nature of their solution sets and this is determined by the nature of the formula. For instance, the solution set for most formulas falls into one of the three following categories: (1) An individual integer, (2) a set (or class) of individual integers, or (3) sets of subsets (or classes of subclasses) of individual integers. The context of the formula determines this classification. Thus, the repeated admonition by Gödel that if one variable is substituted for by another variable, then the variable that is used to make the substitution must have the same sign of type as the original variable, seems superfluous. But, we must keep in mind that Gödel has blurred the distinction between variables and constants. However, Gödel’s admonition is correct when variables represent entire formulas since each formula may have a differing assortment of variables.

It is noteworthy that Gödel’s definition of constants includes the term “0" (zero). Therefore, zero is a constant by his definition, but it is also a sign of type 1 by his definition. One of the expressions a sign of type 1 may assume is a, where a is either 0 or a variable of type 1. But, a variable of type 1 can represent zero so a variable can represent at least one constant from Gödel’s system, i.e., zero, as well as number-signs, which are not constants in Gödel’s system. It is confusing, perhaps purposefully.

On page 5 of Gödel’s paper we learn that the generalization of formula a implies the substitution of formula a. Yet, there is one important difference. The difference is that in substitution the variable that replaces the variable v is explicitly stated while in generalization the variable that replaces variable v is not explicitly stated. As Gödel writes on page five, “$(\forall v \cdot a) \Rightarrow \text{subst } a\ ({}^{v}{}_{c})$.”[6] Now is appropriate time to define two additional terms that will be used throughout the paper. The term “⇒” indicates implies and the term“⇐⇒” indicates implies and is implied by or is equivalent to. These terms belong to a category that Gödel denotes as “abbreviations.”

There are two other ways of writing $(\forall v \cdot a)$ and they are as follows: forall(x, y) where y represents the formula and x represents the variable, the other way is forall(17,y) where a prime number larger than 13 represents the Gödel number of the variable (in this case the Gödel number 17 represents the variable x) and y represents the formula in which x is a variable.

There are two other ways of writing subst a $(^{v}{}_{c})$. The first way is as follows: subst(x, v, y) where x represents the formula, v represents a variable in the formula and y represents a sign of the same type that replaces v in the formula. The second way is as follows: subst(x, 19, number(z)) where x represents the formula, a prime number larger than 13 represents the Gödel number of the variable in the formula (in this case the Gödel number 19 represents the variable y) and number(z) represents the number-sign that replaces the variable y in the formula. Notice that we have a number-sign replacing a variable. To assure ourselves that number(z) is indeed a number-sign, we must refer to subsection 2.4 of Gödel’s paper “Expressing metamathematical concepts.” This section delineates a series of concepts that are relevant to his proof, which Gödel labels as primitive recursive functions.

Gödel’s function 17 states the following:

“17. number(n) = succ(n, seq(1))

number(n) is the number-sign for the number n."[7]

The right side of the formula states that n is the $n^{th}$ successor of zero. The term seq(1) is the Gödel number for zero, although in other instances of the use of Gödel numbers the prefix "seq" is dropped. An example is the formula subst(x, 19, number(z)) that was mentioned a few paragraphs before. In standard mathematics replacing a variable with a number-sign would be equivalent to allowing a constant to replace a variable. Gödel has told us that a variable can only be replaced by a sign of the same type. It is only because of his ambiguous definition of a sign of type 1 that this kind of trickery has the least bit of merit. Gödel wants to have it both ways. He wants a variable to be replaced only by a variable of the same type, and he also wants a variable to be replaced by a constant.

With only a few more details, we should be able to critically analyze, at least partially, Gödel's final numbered formulas, Formulas (15) and (16), from the formal proof of his first incompleteness theorem. Gödel writes on page six, "The proposition, that there are undecidable problems in system P for example reads like this: There are proposition-formulae a, such that neither a nor the negation of a is a provable formula."[8] The proposition-formulas we are about to examine in the final numbered formulas of Gödel's formal proof are: forall(17,r) and ~forall(17,r).

Gödel writes on page 16, “r is a primitive recursive class-sign with the free variable 17.”[9] Therefore, r is a formula with only one free variable x. According to the definition of generalization as it applies to the term forall(17,r) we must replace the free variable x with another free variable. But, according to Gödel, proposition-formulas contain no free variables. He writes on page four, “A formula that contains no free variables (where free variables is interpreted in the usual manner) is called proposition-formula.”[10] Try as he might, Gödel cannot escape from his complex prevarication. The definition of forall (17,r) calls for the substitution of the free variable x with another free variable such as z, but for his proof to succeed he must replace the variable x with a number-sign, i.e., a constant. This is the so because the only kinds of formulas Gödel can prove true or false are those without variables such as: 3 = 3, 4 = 3, 4 + 3 = 7, 4 + 3 = 8, etc.

Below are Gödel’s last two, numbered formulas, Formulas (15) and (16), from his first incompleteness proof. On page 16 Gödel states the following:

“Formula (15)

$\sim$proof $\text{For}_k$ (x, forall (17,r)) $\Rightarrow$ $\text{provable}_k$ subst(r, 17, number(x))

Formula (16)

$\text{proofFor}_k$ (x, forall (17,r)) $\Rightarrow$ $\text{provable}_k$ not(subst(r, 17, number(x)))” [11]

(The subscript “k” refers to any ϖ-consistent primitive recursive class of formulas.)

Formula (15) states that a series of formulas can be grouped together to form a proof and this proof can be denoted by the Gödel number x. This series of formulas is represented by the x on the left side of Formula (15). And, moreover, there is no proof that can be assembled from any ϖ-consistent primitive recursive class of formulas (and denoted by the Gödel number x ) that can prove the proposition-formula forall(17,r). And, all this together implies that the proposition-formula subst(r, 17, number (x)) is provable from a proof assembled from ϖ-consistent primitive recursive class of formulas.

The formula subst(r, 17, number(x)) states that there is a formula r that, as Gödel has previously explained, has one free variable x, which is denoted by the Gödel number 17. And, in the formula r we must substitute the number-sign for the Gödel number x for each free occurrence of the variable x in formula r. The Gödel number x is the Gödel number of the proof on the left side of Formula (15).

The dilemma is that forall(17,r) ⇒ subst(r, 17, number(x)) because of the axiom $(\forall v \cdot a) \Rightarrow$ subst a $({}^{v}{}_{c})$ found on page five of Gödel’s paper. Since Formula (15) states that the inability to prove forall(17,r) implies the ability to prove subst(r, 17, number(x)), a contradictory circumstance is brought about when it is demonstrated that forall(17,r) $\Rightarrow$ subst(r, 17, number(x)). The contradictory circumstance is like the statement, “The inability to prove that dogs have four legs implies the ability to prove that dogs have four limbs.”

There is a way out of the dilemma. If we acknowledge that the axiom $(\forall v \cdot a) \Rightarrow \text{subst } a \left({}^{v}{}_{c}\right)$ is only valid when one variable is substituted for another variable of the same type, and therefore, it is not valid when a number-sign is substituted for a variable. Also, we should acknowledge that the definition of generalization is only valid when one variable is substituted for another, and therefore, it is not valid when a number-sign is substituted for a variable. We know a number-sign is substituted for a variable in the formula subst(r, 17, number(x)) because that fact is explicitly stated. We know a number-sign must be substituted for a variable in the formula forall(17,r) because Gödel has told us that forall(17,r) is a proposition-formula and proposition-formula contain no free variables.

Let us say the variable r represents the formula $x + 3 = 5$. The generalization of r or $(\forall x \cdot r)$ allows us to substitute an arbitrary variable (of the same type) for x. Thus, we have $y + 3 = 5$, $z + 3 = 5$, $a + 3 = 5$, $b + 3 = 5$ . . . etc., none of which contradicts the generalization of r because each formula is unprovable. Therefore, each generalization of the unprovable formula, $x + 3 = 5$, generates another unprovable formula. If we substitute arbitrary number-signs for the variable x, we produce formulas such as, $2 + 3 = 5$, $3 + 3 = 5$, $4 + 3 = 5$, $5 + 3 = 5$ . . . etc., one of which is true(provable) and the rest of which are false(unprovable). They invalidate the generalization of r because a true(provable) statement such as $2 + 3 = 5$ can be generated from an unprovable statement such as $x + 3 = 5$. Therefore, Gödel's use of the axiom $(\forall x \cdot r) \Rightarrow \text{subst}\ r\ ({}^{x}{}_{c})$ when the variable x (on the right side of the formula) is substituted for by a number-sign is invalid, and so his proof is also invalid.

A similar argument can be used to demonstrate that the term subst r $({}^{x}{}_{c})$ is invalid when the variable x is replaced by any specific number-sign. When the variable x in the formula r is replaced by a number-sign, formula r is transformed from a strictly unprovable formula into a formula that is either provable or unprovable depending on the value of the number-sign. Thus, subst r $({}^{x}{}_{c})$ would be invalid because a strictly unprovable formula r would be changed into a conditionally unprovable formula r. Or, stated another way, a strictly unprovable formula would generate a formula that was provable under certain circumstances, for instance $2 + 3 = 5$.

If I was to precisely follow Gödel's system, I would need to rewrite the above equations. For example, $y + 3 = 5$ would become Gödel number 19 +succ(succ(succ(0)))= succ(succ(succ(succ(succ(0))))). However, the results would be the same.

Formula (16) is the negation of Formula (15). Therefore, the arguments made to escape the dilemma posed by Formula (15) can be employed to escape the dilemma posed by Formula (16).

Formula (16) states that a series of formulas can be grouped together to form a proof and this proof can be denoted by the Gödel number x. This series of formulas is represented by the x on the left side of Formula (16). And, moreover, this proof that can be assembled from a ω-consistent primitive recursive class of formulas (and denoted by the Gödel number x ) can prove the formula forall(17,r). And, all this together implies that the formula subst(r, 17, number (x)) is not provable from a proof assembled from ω-consistent primitive recursive class of formulas.

The formula subst(r, 17, number(x)) states that there is a formula r that, as Gödel has previously explained, has one free variable x, which is denoted by the Gödel number 17. And, in the formula r we must substitute the number-sign for the Gödel number x for each free occurrence of the variable x in formula r . The Gödel number x is the Gödel number of the proof on the left side of Formula (16).

Formula (16) makes more sense than Formula (15). In Formula (16) the Gödel number x is the Gödel number of a particular proof. Is Gödel number x in Formula (15) the Gödel number of a particular proof? Or, is it the Gödel number of all the possible proofs that fail to prove the formula forall(17,r)? Is the number of all possible proofs that fail to prove the formula forall(17,r) finite or infinite?

## Part II

We can now turn our attention to the subsection “2.5–Denotability and provability.” This is a short subsection that introduces Theorem V. Theorem V is central to Gödel’s proof of his first incompleteness theorem. Theorem V is directly involved in the formation of Formulas (15) and (16). This centrality is further indicated by Theorem V’s numerical position. In his original paper, Gödel’s first incompleteness theorem is denoted as Theorem VI. Since this is so, it seems extraordinarily odd that Gödel does not provide a formal proof of this theorem and offers instead only, “a sketchy outline of the proof for this theorem.”[12]

On page 13, Gödel states, “The fact that can be expressed vaguely by: Every primitive recursive relation is definable within system P (interpreting that system as to content), will be expressed in the following theorem without referring to the interpretation of formulae of P:

“Theorem V: For every primitive recursive relation $R(x_1,. . . . ,x_n)$ there is a relation sign r (with the free variables $u_1 , . . . , u_n$), such that for each n-tuple $(x_1, \ldots, x_n)$ the following holds:

Formula (3)

$$R(x_1, \ldots, x_n) \Rightarrow \text{provable}(\text{subst}(r, u_1 \ldots u_n, \text{number}(x_1) \ldots \text{number}(x_n)))$$

Formula (4)

$$\sim R(x_1, \ldots, x_n) \Rightarrow \text{provable}(\text{not}(\text{subst}(r, u_1 \ldots u_n, \text{number}(x_1) \ldots \text{number}(x_n))))$$

"We content ourselves with giving a sketchy outline of the proof for this theorem here, since it does not offer any difficulties in principle and is rather cumbersome."[13]

This last statement is reminiscent of Pierre De Fermat's famous claim to have devised a proof of the statement that no equation of the form $x^n + y^n = z^n$ with n greater than 2 has any solutions in positive integers. Fermat claimed to have a proof of this statement, but unfortunately it did not fit into the margin of the work by Diophantus that he was reading at the moment.

If we use a primitive recursive function that Gödel calls the successor function, $\text{succ}(x) = x + 1$, as an example, we will find an error in Formulas (3) and (4). The error is that the terms $\text{number}(x_1)$ and $\text{number}(x_n)$ occurring in Formula (3) are not the same as the terms $\text{number}(x_1)$ and $\text{number}(x_n)$ occurring in Formula (4). This leads to the conclusion that the term number(x) in Formula (15) is not the same as the term number(x) in Formula (16). First, we will discuss the implications of the error, and then we will demonstrate with a simple example that the error does indeed exist.

The formula subst(r, 17, number (x)) is a proposition-formula, that is a formula containing no free variables. The formula forall(17, r) also must be a proposition-formula, according to Gödel. Therefore, the formula forall(17, r) cannot imply the formula subst(r, 17, number(x)) in both Formulas (15) and (16) if the term number(x) is different in both formulas.

Even though the formula subst(r, 17, number(x)) appears to be the same formula in each instance, it is actually two different formulas. They are different in terms of their numerical values and their trueness or falseness. For instance, forall(17, r) could be equivalent to 3 = 3 and subst(r, 17, number(x)) in Formula (15) could be equivalent to 3 = 3 while subst(r, 17, number(x)) in Formula (16) could be equivalent to 4 = 3. Therefore, in this example, forall(17, r) could not imply subst(r, 17, number(x)) as the term occurs in Formula (16). And, it follows, there would be no contradiction in Formula (16) and forall(17, r) would be provable. And, once Formula (16) is determined to be provable and therefore valid, Formula (15), the negation of Formula (16), is invalid. The formulas given in the examples above all utilized equal signs (=), but formulas employing other signs, such as not equal (≠), are not excluded. But, it should be noted the formula x > 7 does not imply the formula x > 8.

The demonstration that the number(x) occurring in Formula (15) is not the same as the number(x) occurring in Formula (16) is important. This is important because it demonstrates that the formula subst(r, 17, number(x)) is not the same in Formulas (15) and (16) even though it appears to be the same. Thus, even if we allow Gödel the trickery of placing variables and constants in the same category, the dilemma he tries to create can be escaped by properly restricting the number of formulas that can be implied by forall(17, r) combined with the correct interpretation of formula subst(r, 17, number(x)). When the formula forall(17, r) is a proposition-formula, as Gödel claims it is in this instance, the number of formulas that it can imply is limited to precisely one formula, itself. It should be pointed out that forall(17, r) as a proposition-formula could imply many identical variations of itself such as 3 = 3 and 2 + 1 = 3. But, that does not allow Gödel's dilemma to reassert itself once we have demonstrated that the term number(x) occurring in the formula subst(r, 17, number(x)), which is located on the right sides of both Formulas (15) and (16), is not the same in each formula.

Now that we have demonstrated how Godel's error in Theorem V undermines his proof of Theorem VI, i.e., the first incompleteness theorem, we should present a specific example of the error. It will be easier to understand an application of Formulas (3) and (4) to a specific primitive recursive function (Gödel's successor function), if we first make a digression for a short discussion of primitive recursive functions.

Primitive recursive functions are collections of functions. Each particular collection of functions originates from one function that serves as a template. In the successor function the term succ(x) = x +1 serves as the template. In many primitive recursive functions, the result of the previous function is inserted in place of the variable of the function under consideration in order to generate a new result. This new result is inserted in place of the variable in the succeeding function and the process continues in an endless repartition. Of course, for the function that serves as the template there is no previous function and hence no previous result to insert in place of the variable. In this case zero is often inserted in the place of the variable of the function that is the template.

For example, let $R \Rightarrow \text{succ}(x) = x + 1$

$x_1$ $\text{succ}(x) = x + 1$

$x_2$ $\text{succ}(\text{succ}(x)) = (x + 1) + 1$

$x_3$ $\text{succ}(\text{succ}(\text{succ}(x))) = ((x + 1) + 1) + 1$

$x_4$ $\text{succ}(\text{succ}(\text{succ}(\text{succ}(x)))) = (((x + 1) + 1) + 1) + 1$

The collection of functions would continue following this pattern without ending. And, $\sim R \Rightarrow \text{succ}(x) \neq x + 1$ would generate a similar never-ending pattern.

$x_1$ $\text{succ}(x) \neq x + 1$

$x_2$ $\qquad \mathrm{succ}(\mathrm{succ}(x)) \neq (x + 1) + 1$

$x_3$ $\qquad \mathrm{succ}(\mathrm{succ}(\mathrm{succ}(x))) \neq ((x + 1) + 1) + 1$

$x_4$ $\qquad \mathrm{succ}(\mathrm{succ}(\mathrm{succ}(\mathrm{succ}(x)))) \neq (((x + 1) + 1) + 1) + 1$

We are now ready to return to Formulas (3) and (4).

Formula(3 )

$$R(x_1, \ldots, x_n) \Rightarrow \mathrm{provable}(\mathrm{subst}(r, u_1 \ldots u_n, \mathrm{number}(x_1) \ldots \mathrm{number}(x_n)))$$

$, \ldots, x_n) \Rightarrow \mathrm{provable}(\mathrm{subst}\ (r, u_1 \ldots u_n, \mathrm{number}(x_1) \ldots \mathrm{number}(x_n)))$

Formula (4)

$$\sim R(x_1, \ldots, x_n) \Rightarrow \mathrm{provable}(\mathrm{not}(\mathrm{subst}(r, u_1 \ldots u_n, \mathrm{number}(x_1) \ldots \mathrm{number}(x_n))))$$

Formula (3) states that R is a collection of primitive recursive formulas. They all originate from a single template. These conditions imply that there is a formula r that has variables $u_1$ through $u_n$, where each of the variables is associated with a specific member of the collection of primitive recursive formulas (as designated by its subscript). And, further, if you substitute for each of the variables $u_1$ through $u_n$ the numeric value of the Gödel number of the formula to which that variable is specifically associated, the formula you generate will be provable.

Formula (4) states that $\sim$R is a collection of primitive recursive formulas. They all originate from the negation of a single template. These conditions imply that there is a formula r that has variables $u_1$ through $u_n$, where each of the variables is associated with a specific member of the collection of primitive recursive formulas (as designated by its subscript). And, further, if you substitute for each of the variables $u_1$ through $u_n$ the numeric value of the Gödel number of the formula to which that variable is specifically associated, the formula you generate will not be provable. The formula r is the same formula in both Formulas (3) and (4). Formula r is designed so that it is provable in Formula(3) and unprovable in Formula(4). Formula r could be designed to fall into any one of three other categories. It could be designed to be provable in both Formulas (3) and (4), or it could be designed to be unprovable in both Formulas (3) and (4), or it could be designed to be unprovable in Formula (3) and provable in Formula (4).

For our example, we will use $R(x_1, x_2)$ and $\sim R(x_1, x_2)$. We will use a simplified method to determine the Gödel number of each of the formulas; a method that ignores parentheses and is designed to produce manageable Gödel numbers. However, the Gödel numbers will only be accurate to the first nine decimal places. In this simplified method each of the following six terms is assigned a Gödel number:

succ . . . 1, x . . . 2, 0 . . . 3, + . . . 4, = . . . 5, and $\neq$ . . . 6. In a formula like a = x + 1 the number-sign for the number 1 is succ(0). Instead of writing a large number that is not a Gödel number this way succ(1,000,000, seq(3)) we will merely write 1,000,000.

The Gödel number of succ(x) = x + 1 is $2.577385992 \times 10^{15}$, which is also number($x_1$) in formula (3).

The Gödel number of succ(succ(x)) = (x + 1) + 1 is

$2.456090619 \times 10^{29}$, which is also number($x_2$) in formula (3).

The formula r $\Longleftrightarrow u_2 \div u_1 = 9.52938608 \times 10^{13}$.

The formula r is provable when the number($x_2$) from formula (3) is substituted for $u_2$ and the number($x_1$) from formula (3) is substituted for $u_1$. The formula is provable because it is true.

The Gödel number of succ(x) $\neq$ x + 1 is $1.288692996 \times 10^{16}$, which is also number($x_1$) in formula (4).

The Gödel number of succ(succ(x)) $\neq$ (x + 1) + 1 is

$1.719263433 \times 10^{30}$, which is also number($x_2$) in formula (4).

The formula r is not provable when the number($x_2$) from formula (4) is substituted for $u_2$ and the number($x_1$) from formula (4) is substituted for $u_1$. The formula is not provable in this instance because it is false; $(1.719263433 \times 10^{30}) \div (1.288692996 \times 10^{16}) \neq 9.52938608 \times 10^{13}$ instead it equals $1.334114051 \times 10^{14}$.

The calculations above demonstrate that the terms number($x_1$) and number($x_2$) occurring in Formula (3) are not the same as the terms number($x_1$) and number($x_2$) occurring in Formula (4). When we examine Formulas (9) and (10) in Part IV, we will discover that the same situation is present. The terms number(x) and number(y) occurring in Formula (9) are not the same as the terms number(x) and number(y) occurring in Formula (10). Furthermore, Formula (9) is generated using Formula (3) as a template, and Formula (10) is generated using Formula (4) as a template. Likewise, when we examine Formulas (15) and (16) in Part IV of this paper, we will see that they use Formulas (9) and (10) as templates, respectively.

We will also see that the term number(x) in Formula (15) is not the same as the term number(x) in Formula (16). The term number(y) is not present in Formulas (15) or (16) because it has been incorporated into formula r. This indicates that the formula r occurring in Formula (15) is not the same as the formula r occurring in Formula (16). The statement that the term number(x) in Formula (15) is not the same as the term number(x) in Formula (16) encapsulates a string of errors. Their significance lies in the fact that they, once again, undermine Gödel's argument that neither forall(17, r) nor its negation ~forall(17, r) is provable.

On page 23, we gave an example where forall(17, r) was represented by 3 = 3 and subst(r, 17, number(x)) from Formula (15) was represented by 3 = 3 and subst(r, 17, number(x)) from Formula (16) was represented by 4 = 3. In this example forall(17, r) was determined to be provable. But, the example could have been constructed to demonstrate that forall(17, r) was not provable and ~forall(17, r) was provable. The point is that either forall(17, r) or its negation, ~forall(17, r), is always provable and that the statement: they are neither provable nor unprovable is false.

## Part III

Subsection "2.4–Expressing metamathematical concepts" is 5 pages in length and consists almost exclusively of formulas. Because of its complexity, it is difficult to give it the kind of in depth appraisal it deserves. On page 9 in the introductory paragraph for subsection 2.5, Gödel briefly summarizes what is to follow, "we will now define a sequence of functions (relations) 1–45, each of which is defined from the preceding ones by the methods given by theorems I through V."[14] There is one additional function, function 46, but as Gödel states, on page 13, ". . . we cannot assert that it is primitive recursive."[15]

Only a select group of functions will be examined in the following discussion.

We begin on page 9 with function 4, and Gödel states:

"4. 0! = 1

$(n + 1)! = (n + 1) \cdot n!$”[16]

The second statement is true. The first statement, $0! = 1$, is both meaningless and true. But, it is true only in an arbitrary fashion. The Mathematics Dictionary defines the word factorial as, “The product of all the positive integers less than or equal to the integer. Factorial n is denoted by the symbol n! E.g., $1! = 1$, $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, and in general, $n! = 1 \cdot 2 \cdot 3 \cdots n$. This definition of factorial leaves the case that occurs when n is zero as meaningless. In order to make certain formulas valid in all cases, factorial zero is arbitrarily defined to be unity. This is the case despite the fact that this is the value of factorial 1.”[17] The key point to note is that to make the formula, $0! = 1$, valid requires the introduction of an arbitrary definition. Arbitrariness is a hallmark of Gödel’s attempt to validate his first incompleteness theorem.

We now jump to function 32, which has no discernable connection with any of the preceding functions. Of the functions, which come after it only functions 35, 38, 39 and 40 employ any of the concepts delineated in function 32. As will become apparent later, these functions employ the concepts described in function 32 literally in name only.

On page 12, Gödel states:

“32. imp(x, y) = or(not(x), y)

and(x, y) = not(or(not(x), not(y)))

equiv(x, y) = and(imp(x, y), imp(y, x))

exits(v, y) = not(forall(v, not(y)))"[18]

The above formulas become increasingly complex. For instance, another way to state equiv(x, y) is as follows: equiv(x, y) = the negation of the negation of this statement: (the term (y) or the negation of the term(x)) or the negation of the negation of this statement: (the term (x) or the negation of the term(y)). Another way to represent it would be as follows: not(or(not(or(not(x), y)), not(or(not(y), x)))). With this representation, precise attention must be paid to each pair of parentheses and the placement of each comma.

There is not any connection between function 32 and the next function 33 other than that they both serve as definitions. There is no attempt made to introduce even superficial connections.

The term type-lift is first introduced by Gödel in subsection "2.1–Definitions." The definition of type-lift presented in function 33 is hampered by the following fact. When discussing the Gödel numbers above Gödel number 13, there are no clearly established boundary lines. Gödel numbers above 13 have no categorical distinctions that represent the various signs of type n that are used to denote all the different kinds of variables. If the Gödel numbers contained clear boundary lines for all the different kinds of variables, then they could be utilized to describe a type-lift.

On page 12, Gödel states:

“33. type Lift(n, x) = argmin $y \leq x^{x\,n} \cdot \forall k \leq \text{length}(x) \cdot$

$\text{item}(k, x) \leq 13 \wedge \text{item}(k, y) = \text{item}(k, x) \vee$

$\text{item}(k, x) > 13 \wedge \text{item}(k, y) = \text{item}(k, x) \cdot \text{prFactor}(1, \text{item}(k, x))^n$

typeLift(n, x) is the nth type-lift of x (if x and typeLift(n, x) are formulae).”[19]

On page 5 Gödel gives a clearer definition of type-lift, “we say that a formula a is a type-lift of another formula b if you can obtain a from b by increasing the type of all variables occurring in b by the same number.”[20]

The term item(k, x) $\leq$13 indicates that k is less than or equal to Gödel number 13. Therefore, it is a constant, and as such it does not qualify as a formula. The term item(k, x) >13 indicates that k could be either a variable or a formula.

The definition of type-lift presented in function 33 is not a model of lucidity. But, more importantly the relationship between function 33 and the next function, 34, is completely forced.

It is revealing that Gödel was compelled to employ an introductory sentence in an attempt to bridge the gap. Function 34 along with its introductory sentence state the following: “There are three specific numbers corresponding to the axioms I, 1 to 3, which we will denote by $pa_1$, $pa_2$, $pa_3$, and we define:

“34. PeanoAxiom(x) $\Longleftrightarrow$ (x = $pa_1 \vee x = pa_2 \vee x = pa_3$)”[21]

Function 34 has no primitive recursive relationship to function 33. Yet, that non-relationship masquerades as a relationship between primitive recursive functions. A calculated attempt is made to demonstrate that $pa_3$ is a type-lift of $pa_2$ and that $pa_2$ is a type-lift of $pa_1$. This is not the case. The actual axiom represented by the term $pa_3$ is not a type-lift of the actual axiom represented by the term $pa_2$, and the actual axiom represented by the term $pa_2$ is not a type-lift of the actual axiom represented by the term $pa_1$.

The terms themselves cannot be type-lifts because type-lifts can only involve formula and these terms do not fit the requirements for elementary formulas; they only superficially resemble elementary formulas. On page 4 Gödel states, “We call combinations of signs of the form a(b), where b is a sign of type n and a a sign of type n + 1, elementary formulae.”[22] If the terms themselves were elementary formulas in which the third formula was a type-lift of the second formula and the second formula was a type-lift of the first formula, they would appear in this kind of formulation: $x = p_2(a_1) \vee x = p_3(a_2) \vee x = p_4(a_3)$.

We will now discuss functions 38, 39 and 40. They all employ terminology from function 32, and the terminology from function 32 is employed ambiguously. Function 35 also ambiguously employs terminology from function 32. The pattern the ambiguous terminology follows in function 35 is the same as the pattern that will be delineated with the following three functions.

Function 38 is very densely constructed. It seems to offer a definition (with more clarity?) of the relation between signs of type1 and variables of type1, and it does employ the term imp(x, y), which is defined in function 32. Function 38 begins with the word the quantor. Quantor may be derived from the word quantic. The Mathematics Dictionary defines quantic as, “A rational integral homogeneous function of two or more variables, or a homogeneous algebraic polynomial in two or more variables. . . .They are classified as binary, ternary, quaternary, etc., according as they contain two, three, four, etc., variables.”[23]

Function 38 states the following:

“38. quantor 1 Axiom(x)$\Longleftrightarrow \exists$ v, y, z, n $\leq$ x $\cdot$
vtype(n, v) $\wedge$ stype(n, z) $\wedge$ is Fm(y) $\wedge$ quantor 1 Axiom
Condition(z, y, v) $\wedge$ x = imp(forall(v, y), subst(y, v, z))
x is a formula obtained by substitution from the axiom schema III,1.”[24]

On page 5 Gödel presents the axiom schema III,1 as the following: “1. ($\forall$ v $\cdot$ a) $\Rightarrow$ subst a$({}^{v}{}_{a})$.”[25] If we interpreted the formula x = imp(forall(v, y), subst(y, v, z)) from function 38 as stating x is equivalent to the statement: forall(v, y) implies subst(y, v, z), there would be no conflict between axiom schema III,1 and formula x. But, function 32 reminds us that the term imp(x, y) does not denote implies, instead it denotes: imp(x, y) = or(not(x), y). Therefore, formula x states the following: x is equivalent to, either the negation of forall(v, y) or the subst(y, v, z). This more rigorous interpretation is the correct interpretation, and it is not related to axiom schema III,1. It does resemble function 43, which is a definition of the term immediate consequence. Function 38 employs the terminology from function 32 in (abbreviated) name only.

The preceding ambiguity may serve a purpose. It may distract us from considering that it is still not precisely clear what the relationship is between the term, vtype(n, v) when n = 1, and the term, sign of type n when n = 1. A variable of type 1 is a variable that represents an unknown individual natural number including zero. What does the term stype(n, z) denote when n = 1? A sign of type 1 is either a specific natural number including zero or it is a variable of type 1 to which we add a specific natural number including zero.

Thus, the term subst(y, v, z) still presents the same confusion discussed previously, if vtype(n, v) is variable of type 1 and stype(n, z) is a sign of type 1. There are three terms denoted by sign of type(n, z) when n = 1, any of which can be substituted for the variable vtype(n, v) when n = 1, in the formula y, yet only one is correct. The three terms denoted by a stype(n, z) when n = 1 are as follows: (1) a specific natural number such as five, (2) the term x + (any specific natural number including zero) such as the term x +7, and (3) the term x + 0. Only the third term, x + 0, can be correctly substituted for a variable of type 1, yet Gödel allows all three terms to be substituted for a variable of type 1.

Function 39 has the same kind of ambiguous use of the terminology, drawn from function 32, as function 38 does, and there are other similarities to function 38, as well.

Function 39 states the following:

“39. quantor 2 Axiom(x) $\Longleftrightarrow \exists$ v, q, p $\le$ x $\cdot$

is Var(v) $\wedge$ is Fm(p) $\wedge$ $\sim$free(v, p) $\wedge$ is Fm(q) $\wedge$

x = imp(forall(v, or(p, q), or(p, forall(v, q)))

x is a formula obtained by substitution from the axiom schema III, 2.”[26]

On page 5 Gödel presents the axiom schema III, 2 as the following: “($\forall$ v $\cdot$ b $\vee$ a) $\Rightarrow$ (b $\vee$ $\forall$ v $\cdot$ a).”[27] If we interpreted formula x = imp(forall(v, or(p, q), or(p, forall(v, q))) from function 39 as stating x is equal to the statement, forall(v, or(p, q)) implies or(p, forall(v, q)), there would be no conflict between axiom schema III,2 and formula x. But, again, function 32 reminds us that the term imp(x, y) does not denote implies; it denotes this: imp(x, y) = or(not(x), y). Therefore, formula x states the following: x is equivalent to, either the negation of forall(v, or(p, q)) or or(p, forall(v, q)). There is another way of representing formula x that requires strict attention to the pairing of parentheses and the placement of commas. It is as follows: x = or(not(forall(v, or(p, q))), or(p, forall(v, q))). These two representations are not related to axiom schema III,2.

Function 40 has the same type of ambiguity, again drawn from function 32, as functions 39 and 38. It also generates a very long formula, which adds to the confusion.

Function 40 states the following:

“40. redu Axiom(x) $\Longleftrightarrow$ $\exists$ u, y, n $\leq$ x $\cdot$
vtype(n, v) $\wedge$ vtype(n + 1, u) $\wedge$ $\sim$free(u, y) $\wedge$ is Fm(y) $\wedge$
x = exists(u, forall(v, equiv(seq(u) $\circ$ paren(seqv)), y)))
x is a formula obtained by substitution from the axiom schema IV,1.”[28]

On page 5, Gödel presents the axiom schema IV, 1 as the following: “$\exists$u $\cdot$ $\forall$v $\cdot$ (u(v) $\Longleftrightarrow$ a).”[29] Function 32 provides us with the definitions of the terms exists(x, y) and equiv(x, y). The term exists(x, y) = not(forall(v, not(y))), and the term equiv(x, y) = and(imp(x, y), imp(y, x)). Only if we completely disregard those definitions and give the terms exists(x, y) and equiv(x, y) new definitions, will formula x be equivalent to axiom schema IV,1. Under the conditions of the new definitions, exists(x, y) = there exists an x such that y and equiv(x, y) = (x ⇐⇒y), which denotes x is equivalent to y. If we utilize these new definitions formula x is equivalent to axiom schema IV,1. If we do not utilize these new and entirely different definitions, formula x becomes very complex, as the following representation shows: x = not(forall(u, not(forall(v, not(or(not(or(not(u(v), y)), not(or(not(y), u(v))))))))). And, the formula x just denoted does not represent axiom schema IV,1.

If Gödel is sending a message with function 40, as well as with other functions in subsection 2.4, the message is clear. The message is that in his description of system P his definitions must be allowed to have a great deal of elasticity. They must be allowed to have an elasticity that defies common sense.

Many of the 46 functions in subsection "2.4 Expressing metamathematical concepts" are reasonable. Many other functions do not reach that level. Gödel does not accomplish his task of defining, "a sequence of functions 1–45, each of which is defined from the preceding ones by the methods given by theorems I through IV."[30]

## Part IV

The formal proof of Gödel's first incompleteness theorem, which is denoted as Theorem VI in his original paper, begins on page 15 and ends on page 16 of Hirzel's translation.  The following exposition becomes quite involved.

Although, the following exposition is quite involved, the first portion of it, which is concerned with the six formulas that begin the formal proof, can be countered  by the simple expedient of claiming that the six formulas (5, 6, 6.1, 7, 8, 8.1) are all axioms.  This does not seem to be the case, though, because the axioms for the system P are listed on pages 5 and 6.  The axioms are listed again as axioms that are also primitive recursive functions.  These axioms/primitive recursive functions are functions 34 through 42, which are listed on  pages 12 and 13.  The six formulas that launch the formal proof are not among the axioms listed for system P.  Primitive recursive functions 44, 45 and 46 are the nearly identical counterparts of the three formulas that launch the formal proof, i.e., (Formulas 5, 6, 6.1).  If Gödel had intended that the primitive recursive functions 44, 45 and 46 should be axioms, it is odd he did not include them with the list of axioms that just precedes them, i.e., axioms/primitive recursive functions 34 through 42.

Also, it seems that Gödel did not intend that the six formulas, which launch the formal proof, should be definitions because the definitions are listed in subsection “2.1 Definitions” and these six formulas are not among them. Perhaps, he considers them metamathematical concepts because subsection “2.4–Expressing metamathematical concepts” is where the counterparts to the three formulas that launch the formal proof are listed. The text of the formal proof seems to infer that they are definitions of a sort. Perhaps, regarding them as definitions derived from analogous metamathematical concepts is the best way to understand them.

The following exposition gives an added flavor to Gödel’s ambiguity. The six formulas, with which the formal proof begins, share the similarity of being definitions of a sort. They will be discussed in section A. The remaining eight numbered formulas of the formal proof share a kind of stepwise relationship. They will be discussed in section B.

### Section A

The formal proof’s first formula, which is denoted by the number (5), states the following:

“is Proof Figure$_k$(x) $\Longleftrightarrow$ ($\forall n \leq$ length(x) $\cdot$ is Axiom(item(n, x)) $\vee$ (item(n, x) $\in$ $\kappa$) $\vee$ $\exists\, 0 < p, q < n$ $\cdot$ immed Conseq (item(n, x), item(p, x), item(q, x))) $\wedge$ length(x)>0”[31]

The formula states x is a proof figure within any ω-consistent primitive recursive class of formulas. The proof figure must be a finite sequence of formulas. Furthermore, each formula must meet at least one of the three following criteria: (1) the formula is an axiom, (2) the formula is the immediate consequence of the two preceding formulas, or (3) the formula is a member of any ω-consistent primitive recursive class of formulas.

Does Formula (5) meet the requirements, which the formula itself stipulates, for being a valid formula in a proof? Formula (5) is not the immediate consequence of the two preceding formulas in the proof because it is the first formula of the proof. It is not an axiom. Therefore, it must be a member of a ω-consistent primitive recursive class of formulas, but this is not the case. It is similar to primitive recursive concept 44 in subsection "2.4–Expressing metamathematical concepts," but it is not exactly the same. Gödel reminds us twice that the concepts 1 through 45 in subsection 2.4 are primitive recursive functions. (The term "function 1, 2, 3, etc." used in Part III of this paper is interchangeable with the term "primitive recursive concept 1, 2, 3, etc." or "concept 1, 2, 3, etc." used in Part IV of this paper. The nomenclature is employed because the terms serve a dual role as both functions and concepts.)

There are two differences between Formula (5) and concept 44. The first difference is the introduction of the subscript "k," and the second and most significant difference is that concept 44 does not include the phrase: or a member of a ω-consistent primitive recursive class of formulas. This difference, by itself, should disqualify Formula(5) from being considered a member of any primitive recursive class of formulas that has been described by Gödel so far. And, thus, since it is neither an axiom nor the immediate consequence of the two preceding formulas it should be disqualified as a valid formula for a proof.

Once again, Gödel has taken a straightforward mathematical concept and altered it so that it has become ambiguous. In this case, it is the concept of valid formulas for a proof. Gödel states in primitive recursive concept 44:

“is Proof Figure(x) $\Longleftrightarrow (\forall\, 0< n \leq \text{length}(x) \cdot \text{is Axiom}(\text{item}(n, x)) \vee \exists\, 0< p, q < n \cdot \text{immConseq}(\text{item}(n, x), \text{item}(p, x), \text{item}(q, x))) \wedge \text{length}(x) >0$

x is a proof figure (a finite sequence of formulae, each of which is either an axiom or the immediate consequence of the two preceding ones).”[32]

This is a straightforward formulation of a mathematical proof though it may be too restrictive a formulation. Then in Formula (5) Gödel alters this straightforward concept. He makes an addition to the type of formulas that can be used in a proof. Besides formulas that are either the immediate consequence of the two previous formulas or axioms, he adds a third type of formula, any formula that is a member of any ω-consistent primitive recursive class of formulas.  With this alteration, the formulation for a mathematical proof may be too inclusive.  Gödel cannot find the proper balance.

Gödel doesn’t give any justification for altering the criterion for valid formulas of a mathematical proof.  Why would the criterion for deciding which formulas constitute a valid proof be subject to such an alteration? Furthermore, he doesn’t provide any justification that Formula (5) is a valid formula for a proof.   Even though with the introduction of Formula (5), Gödel changes the definition of a proof in a way that increases the kinds of formulas that are  acceptable in the construction of valid proofs.

Also, the legitimacy is highly suspect of Gödel’s penchant for claiming all sorts of formulas are primitive recursive formulas.  All this evidence leads to the conclusion that Formula (5) is not a valid formula for a proof.

The formal proof’s second formula, which is denoted by the number (6), states the following:

“proof For$_\kappa$(x, y) $\Longleftrightarrow$ is Proof Figure$_\kappa$(x) $\wedge$ item(length(x), x) = y.”[33]

Formula (6) is nearly identical with primitive recursive concept 45, which states the following:

“proof For(x, y) $\Longleftrightarrow$ is Proof  Figure(x) $\wedge$ item(length(x), x) = y
x  is a proof for the formula y.”[34]

The only difference between the two formulas is that Formula (6) has the subscript “κ” in two places. The subscript “k” denotes the phrase: in any ω-consistent primitive recursive class of formulas. Since concept 45 is true for all classes of formulas, at least according to Gödel, restricting its range to the primitive recursive class of formulas by the introduction of the subscript “k” in order to produce Formula (6) is justified. The justification for Formula (6) is that it is very similar to a formula that is a member of a ω-consistent primitivie recursive class of formulas. It cannot be justified any other way. Formula (6) is not an axiom, and since it is the second formula of the formal proof it cannot be the immediate consequence of the two preceding formulas.

The formal proof’s third formula, which is denoted by the number (6.1), states the following:

“$\text{provable}_{\kappa}(x) \Longleftrightarrow \exists y \cdot \text{proof For}_{\kappa}(y, x)$.”[35]

Formula (6.1) is nearly identical with concept 46, which states the following:

“$\text{provable}(x) \Longleftrightarrow \exists y \cdot \text{proof For}(y, x)$

$x$ is a provable formula (provable($x$) is the only one among the concepts 1-46 for which we cannot assert that it is primitive recursive).”[36]

Again, the only difference between Formula (6.1) and concept 46 is that Formula (6.1) has the subscript “κ” in two places. Gödel states that he cannot assert that concept 46 is primitive recursive. Yet, when he twice introduces the subscript “k” into concept 46 in order to produce Formula (6.1) that is precisely the assertion he is making, since the subscript “k” denotes the phrase: in any ω-consistent primitive recursive class of formulas. Or, perhaps, he is saying the concept represented by Formula (6.1) is true for primitive recursive classes of formulas even though it cannot be asserted the formula itself is primitive recursive. If so Formula (6.1) could be an example of Gödel’s incompleteness theorem, unless it is disqualified for being a non-arithmetical statement or for some other reason. Gödel may be suggesting that Formula (6.1) is a statement constructed within a system of primitive recursive formulas that is true, yet its nearly identical counterpart, function 46, cannot be declared a member of a system of primitive recursive formulas. The latter sentiment rings resoundingly hollow since Part III of this paper documented that Gödel considered formulas to be primitive recursive for the slightest of reasons. In fact, Gödel considered certain formulas to be primitive recursive when there was no evidence to support the claim. It is not surprising to discover contradictions in Gödel’s claims. Gödel makes the claim that since the right side of a certain formula is primitive recursive then the left side is also primitive recursive when he states, on page 15, “Since proofFor$_k$(x, y) (by (6), (5)) and subst(y, 19, number(y)) (by definitions 17, 31) are primitive recursive, so is Q(x, y).”[37] Using this precept, the claim can be made that concept 46 is primitive recursive since the right side of concept 46, $\exists y \cdot$ proofFor(y, x), is very similar to the left side of primitive recursive concept 45, proofFor(x, y). This may explain why in concept 46 Gödel reversed his convention that x represents the proof and y represents the provable formula. This reversal makes it more difficult to apply the precept stated above. It is more evidence of the arbitrariness of Gödel's method.

Formula (6.1) is not an axiom. And, Gödel tells us that its counterpart, concept 46, is not a member of his primitive recursive concepts. Therefore, for Formula (6.1) to be a valid formula in the proof it must be the immediate consequence of the two preceding formulas, Formulas (5) and (6). Here is the catch. The counterparts of Formulas (5) and (6) are primitive recursive concepts 44 and 45. These two concepts, according to Gödel, are not related in a primitive recursive manner to concept 46. The methods for determining a primitive recursive relationship include methods very similar to those for determining an immediate consequence relationship. It is unlikely Formula (6.1) will be the immediate consequence of Formulas (5) and (6) when the counterparts of Formulas (5) and (6) do not have a primitive recursive relationship with the counterpart of Formula (6.1).

Yet, for Formula (6.1) to be a valid formula for the proof, we must demonstrate that Formula (6.1) is the immediate consequence of Formulas (5) and (6). Gödel states in primitive recursive concept 43:

“$\text{imm Conseq}(x, y, z) \Longleftrightarrow y = \text{imp}(z, x) \vee \exists v \leq x \cdot \text{is Var}(v) \wedge x = \text{forall}(v, y)$

$x$ is an immediate consequence of $y$ and $z$.”[38]

The formula states that formula $x$ is the immediate consequence of formulas $y$ and $z$. Furthermore, there are two forms formula $y$ can assume. Formula $y$ is either the negation of formula $z$ or formula $y$ equals formula $x$. If formula $y$ is the negation of formula $z$ then formula $x$ is the generalization of formula $y$, which can be written $x = \text{forall}(v, y)$.

Formula (6.1) is not the immediate consequence of the two preceding formulas when the primitive recursive concept 43, which is the definition of immediate consequence, is taken into account. This notion is given added credence by the fact that concept 46, the nearly identical counterpart of Formula (6.1), cannot be obtained from primitive recursive concepts 44 and 45, the nearly identical counterparts of Formulas (5) and (6), respectively.

The formal proof’s fourth formula, which is denoted by the number (7), states the following:

“$\forall x \cdot (\text{provable}_\kappa(x) \Longleftrightarrow x \in \text{Conseq}(\kappa)$”[39]

The formula states that for any formula denoted by $x$ that is provable in any ω-consistent primitive recursive class of formulas, there is the certain implication that the formula $x$ is a member of the smallest set of formulas that contain all the formulas and all the axioms of that particular primitive recursive class of formulas, and the reverse also holds true.

Formula (7) is not the immediate consequence of Formulas (6) and (6.1), and it is not an axiom. Also, it is not a member of any ω-consistent primitive recursive class of formulas enumerated by Gödel in his paper. It does have the character of a definition, but it is not listed as such.

The formal proof’s fifth formula, which is denoted by the number (8), states the following:

“$\forall x \cdot (\text{provable}(x) \Rightarrow \text{provable}_{\kappa}(x))$.”[40]

The formula states that any provable formula x is also a provable formula x in any ω-consistent primitive recursive class of formulas. But, the term “⇒” indicates that the reverse need not hold true.

By the use of the term “⇒” this formula acknowledges that Formula (5) is different from the primitive recursive concept 44. It also acknowledges, by the use of the term “⇒,” that Formula (5) is less restrictive in its definition of what are valid formulas for a proof than is primitive recursive concept 44. If Formula (5) and concept 44 were equally restrictive, the term “⇐⇒” would be used in Formula (8). Like the preceding four formulas, it seems reasonable. It is not an axiom or the immediate consequence of the two preceding formulas. In fact, it is different from all the preceding formulas in the formal proof. They all concern themselves with ω-consistent primitive recursive classes of formulas, and the term provable(x) does not. To understand the meaning of provable(x) we must refer to primitive recursive concepts 46, 45 and 44. To understand the meaning of $\text{provable}_\kappa(x)$ we must refer to Formulas (6.1), (6) and (5). Only a thorough analysis of primitive recursive concepts 46, 45 and 44 plus a thorough analysis of Formulas (6.1), (6) and (5) could lead to the derivation of Formula (8).

The formal proof's sixth formula, which is denoted by the number (8.1), states the following:

“$Q(x, y) \Longleftrightarrow \sim(\text{proof For}_\kappa(x, \text{subst}(y, 19, \text{number }(y))))$.”[41]

The formula states that a series of formula have been organized to form a proof, and that proof is denoted by x. The proof denoted by x does not prove the formula subst(y, 19, number (y)). The term Q(x, y) denotes that the formula is primitive recursive. It is unclear why the term $R(x_1, x_2)$ is not used to denote that the formula is a primitive recursive relation since in Theorem V the general term for a primitive recursive relation is $R(x_1, \ldots, x_n)$. Perhaps, it is to obscure one of the errors that is present in the next two formulas. The formula is introduced by Gödel with these words, “Now we define the relation.”[42] Thus, it has the characteristics of a definition. In the text Gödel makes the assertion that Formula (8.1) is a member of a primitive recursive class of formulas. Gödel makes the claim that if the right side of a formula is primitive recursive then the left side is also primitive recursive. He states that since the terms proofFor$_{k}$(x, y) and subst(y, 19, number(y)), which are the “building blocks” for the right side of Formula (8.1), are primitive recursive, then the term Q(x, y), which is the left side of Formula (8.1), is also primitive recursive.

## Section B

The formal proof’s seventh and eighth formulas, which are denoted by the numbers (9) and (10), are derived by applying Theorem V to Formula (8.1). Gödel makes this clear in the text of the formal proof. He states, “According to theorem V we hence have a relation sign q (with the free variables 17, 19) such that the following holds:

Formula (9)

~ proof For$_{\kappa}$(x, subst(y, 19, number(y))) ⇒ provable$_{\kappa}$(subst(q, 17 19, number(x) number(y)))

Formula(10)

proof For$_{\kappa}$(x, subst(y, 19, number(y))) ⇒ provable$_{\kappa}$(not(subst(q, 17 19, number(x) number(y))))”[43]

There are two errors in Formulas (9) and (10). The first error was pointed out in Part II. The terms number(x) and number(y) that occur on the right side of Formula(9) are not the same as the terms number(x) and number(y) that occur on the right side of Formula (10). The second error is that the term number(y) that occurs on the right sides of both Formulas (9) and (10) should be replaced with the term number(subst(y, 19, number(y))). This is so because the left side of Formula (9) is not $\sim$proof For$_{\kappa}$(x, y), instead it is $\sim$proof For$_{\kappa}$(x, subst(y, 19, number(y))). And, likewise, the left side of Formula (10) is not proof For$_{\kappa}$(x, y), instead it is proof For$_{\kappa}$(x, subst(y, 19, number(y))). Even with this correction the term number(subst(y, 19, number(y))) that should occur on the right side of Formula (9) will be different from the term number(subst(y, 19, number(y))) that should occur on the right side of Formula (10).

The formula subst(y, 19, number(y)) occurs on the right side of Formula (8.1) and on the left sides of Formulas (9) and (10). In all three instances, the term number(y) from the formula subst(y, 19, number(y)) refers to the number-sign value (numeric value) of the Gödel number of formula y. Formula y is represented by the y that occurs within the formula subst(y, 19, number(y)) and not the y that occurs in the term Q(x, y). In Formulas (9) and (10) when the term number(y) occurs on the right sides of the formulas, it is incorrect, since it refers to the number-sign value of the Gödel number of formula y. It is undeniable that number(y) must refer to the number-sign value of the Gödel number of formula y to be consistent with its usage in Formula (8.1) and its usage on the left sides of Formulas (9) and (10). The quantity within the parentheses in the term number(y), occurring on the right sides of Formulas (9) and (10), should refer to the entire formula subst(y, 19, number(y)), not merely the formula y. Therefore, to correct this error, the term number(y) should be replaced by a term that refers to the number-sign value of the Gödel number of the formula subst(y, 19, number(y)), which is to say number(subst(y, 19, number(y))). The confusion arises because the y in the term Q(x, y) from Formula (8.1) does indeed refer to the formula subst(y, 19, number(y)). If the term $R(x_1, x_2)$ had been used instead of the term Q(x, y), perhaps, some of the confusion could have been avoided. This confusion will follow us for the rest of the formal proof.

The formal proof's ninth and tenth formulas, which are denoted by the numbers (11) and (12), respectively, result from two different versions of the operation described in the definition of the generalization of an elementary formula. In this case, the formula that undergoes two different versions of generalization is formula q. Formula q occurs on the right sides of both Formulas (9) and (10). Thus, Formulas (11) and (12) are in an informal way the immediate consequence of a portion of the two preceding formulas. Formula (11) states the following:

"p = forall(17, q)."[44]

Formula (12) states the following:

“r = subst(q, 19, number(p)).”[45]

Notice the two different versions of generalization: the “forall” version and the “subst” version. Notice, also, that in Formula (12) the number-sign, i.e., number(p), which is to be substituted for the variable y, i.e., Gödel number 19 in formula q, is the number-sign for the Gödel number of formula p.

The Formulas (11) and (12) are characterized by Gödel as class signs, in statements occurring on the bottom of page 15 and the top of page 16, respectively. However, Gödel’s characterization of each of the formulas as a class-sign is in error. A class-sign is a formula with one free variable. For reasons we will explore in the discussion of Formula (14), these two formulas, Formulas (11) and (12), cannot be class signs.

The formal proof’s eleventh formula, which is denoted by the number (13), consists of three equations. The first is the generalization of the formula, p = forall(17, q). The formula p is itself a generalization of the formula q. The second equation is the result of applying three axioms to formula p. Once these operations are accomplished, it is apparent that a formula within the formula derived from applying the three axioms to formula p is equivalent to formula r, and thus the third equation is generated. The application of the axioms seems forced. Formula (13) states the following:

“subst(p, 19, number(p)) = subst(forall(17, q), 19, number(p))
= forall(17, subst(q, 19, number(p)))
= forall(17, r) ”[46]

The explanation for the three equations is complex. However, the explanation for the first equation is straightforward. The “subst” version of the operation for generalization of formula p is subst(p, 19, number(p)). Since p = forall(17, q), the term forall(17, q) is a legitimate replacement for formula p and thus the right side of the first equation is generated. Thus, this first equation is informally the immediate consequence of Formula (11).

The generation of the second equation is quite involed. It requires the use of three axioms from section “2.1 Definitions.” The first axiom is axiom II,1, $p \vee p \Rightarrow p$. The second axiom is axiom III,1, $(\forall v \cdot a) \Rightarrow$ subst a $({}^{v}_{c})$. The third axiom is axiom III,2, is $(\forall v \cdot b \vee a) \Rightarrow (b \vee \forall v \cdot a)$. Gödel states that for axioms III,1 and III,2 we should insert an arbitrary formula for a, an arbitrary variable for v, any formula where v does not occur free for b, and for c a sign of the same type as v.

We shall begin with the axiom $(\forall v \cdot b \vee a) \Rightarrow (b \vee \forall v \cdot a)$. We insert forall(17, q) for b and forall(19, q) for a. This gives us ($\forall$v $\cdot$ forall(17, q) $\vee$ forall(19, q) $\Rightarrow$ (forall(17, q) $\vee$ $\forall$v $\cdot$ forall(19, q). Next, we apply axiom $p \vee p \Rightarrow p$ to both the left side and the right side of the previous formula. With regard to the left side of the formula, we apparently judge that $\text{forall}(17, q) \vee \text{forall}(19, q) \Rightarrow \text{forall}(17, q)$. This gives us $\forall v \cdot \text{forall}(17, q)$ for the left side of the formula. With regard to the right side of the formula, we apparently judge that $\text{forall}(17, q) \vee \forall v \cdot \text{forall}(19, q) \Rightarrow \forall v \cdot \text{forall}(19, q)$. This gives us $\forall v \cdot \text{forall}(19, q)$ for the right side of the formula. Combining the left and right sides of the formula gives us $\forall v \cdot \text{forall}(17, q) \Rightarrow \forall v \cdot \text{forall}(19, q)$.

Now, we apply the axiom $(\forall v \cdot a) \Rightarrow \text{subst } a\ (^v_c)$ to the left side of the formula. We insert forall(17, q) for a. We insert y for the variable v, and we insert number(p) for c. This gives us $(\forall y \cdot \text{forall}(17, q)) \Rightarrow \text{subst}(\text{forall}(17, q), y, \text{number}(p))$. Substituting the Gödel number 19 for the variable y gives us subst(forall(17, q), 19, number(p)) for the left side of the formula, which is the right side of the first equation.

Now we apply the same axiom, $(\forall v \cdot a) \Rightarrow \text{subst } a\ (^v_c)$, to a portion of the right side of the same formula, $\forall v \cdot \text{forall}(17, q) \Rightarrow \forall v \cdot \text{forall}(19, q)$. Forall(19, q) is rewritten as $(\forall y \cdot q)$. Applying the axiom gives us $(\forall y \cdot q) \Rightarrow \text{subst}(q, y, \text{number}(p))$. Substituting the Gödel number 19 for the variable y gives us subst(q, 19, number(p)). The entire right side of the formula is $\forall v \cdot \text{subst}(q, 19, \text{number}(p))$. For the variable v in the term $\forall v$ we insert the variable x, but in the form of its Gödel number, which is 17. We rewrite the term $\forall 17 \cdot$ as forall(17, so that the right side of the formula is forall(17, subst(q, 19, number(p))), which is the second equation. Thus, the right side of the first equation implies the second equation. In other words this formula $\forall v \cdot$ forall(17, q) $\Rightarrow \forall v \cdot$ forall(19, q) was changed into this formula subst(forall(17, q), 19, number(p)) $\Rightarrow$ forall(17, subst(q, 19, number(p)) through the use of axiom III,1. This does not mean that the first equation equals the second equation. This distinction will become very important when we analyze Formulas (15) and (16). For Gödel's proof to succeed it is critical that subst(p, 19, number(p)) = forall(17, r), but this is not the case. It is also critical for the success of his proof that formula p or forall(17, q) = subst(p, 19, number(p)), but again this is not the case. When we apply axiom III,1, which is $\forall v \cdot a \Rightarrow$ subst a $({}^{v}_{c})$ to the term forall(17, q) we generate this formula $\forall 19 \cdot$ forall(17, q) $\Rightarrow$ subst(forall(17, q), 19, number(forall(17, q))). The term forall(17, q) occurs twice on the right side of the previous formula. If we replace it on both occasions with formula p, this gives us $\forall v \cdot$ forall(17, q) $\Rightarrow$ subst(p, 19, number(p)). However, if we accept Gödel's contention that formula p = forall(17, q) has only one free variable, which is y (Gödel number 19), when we replace that free variable first with number(p) and then with number(forall(17, q) and compare the results, we can encounter difficulties. Let us say number(p) = 20 and number(forall(17, q) = 10. If formula q $\Longleftrightarrow$ y + 3 = 13, then $\forall v \cdot$ forall(17, q) $\Rightarrow$ 20 + 3 = 13 when number(p) replaces the variable y , but $\forall v \cdot$ forall(17, q) $\Rightarrow$ 10 + 3 = 13 when number(forall(17, r) replaces the variable y, thus a contradiction is introduced to the derivation. This is the same difficulty we encountered before in Part I of this paper when constants were substitutes for variables.

The formula $\forall v \cdot$ forall(17, q) $\Rightarrow$ subst(p, 19, number(p)) is not an equality as Gödel requires, and the term $\forall 19 \cdot$ is not in the equation forall(17, q) = subst(p, 19, number(p)).

The second equation, forall(17, subst(q, 19, number(p))), is not the immediate consequence of the first equation; it is an axiom. This is so because, according to Gödel, every formula obtained by using the axioms in section “2.1 Definitions” is itself an axiom.

The third equation is generated by substituting formula r for subst(q, 19, number(p)) in the second equation. This gives us forall(17, r). Thus, the third equation is the generalization of the second equation.

The formal proof’s twelfth formula, which is denoted by the number (14), is similar to Formula (13) in that the operation defined as generalization is employed. Also, a formula within a formula is revealed to be equivalent to formula r. However, the formula thus revealed is derived from generalization, not from the application of axioms. The right side of Formula (14) is the generalization of Formula (9) with some very important caveats. As a reminder

Formula (9) is the following:

$\sim$ proof For$_{\kappa}$(x, subst(y, 19, number(y))) $\Rightarrow$ provable$_{\kappa}$(subst(q, 17 19, number(x) number(y)))

Formula (9) undergoes the operation of generalization, but it is difficult to recognize that it does so, because the entire left side of Formula (9) is deleted from Formula (14). The deleted left side of Formula(9) undergoes the generalization operation, which is the substitution of formula p for formula subst(y, 19, number(y)). And, also, the term provable$_{\kappa}$ from the right side of Formula (9) is deleted from Formula (14). The left side of Formula (14) is the result of the generalization of Formula (9) with the two deletions mentioned above. Formula (14) states the following:

"subst(q, 17 19, number(x) number(p)) = subst(r, 17, number(x))"[47]

To reiterate, the left side of Formula (14) is derived from the generalization of Formula(9), which in this instance involves substituting the formula p, i.e., Formula (11), p = forall(17, q), for the formula subst(y, 19, number(y) located on the left side of Formula (9). Then we must selectively delete the portions that were mentioned above to obtain the left side of formula (14). The term number(p) replaces the term number(y) because of the substitution of formula p for formula subst(y, 19, number(y)).

The formula, subst(y, 19, number(y)), from the left side of Formula (9) is, according to Gödel, a proposition-formula, and proposition-formulas have no free variables. Thus, it is a sign of type 1. According to the definition of generalization it can only be substituted for by a sign of the same type, i.e., a sign of type 1. The substitute for the formula, subst(y, 19, number(y)), is the formula p = forall(17, q). On page 15, Gödel states, "p is a class-sign with the free variable 19."[48] The free variable 19 (Gödel number 19) is the free variable y. The substitution of the class-sign, p = forall(17, q), which has one free variable for the proposition-formula, subst(y, 19, number(y)), which has no free variables, is legitimate because, according to Gödel's ambiguous definitions, they are both signs of type 1. Although, it may seem strange, it does not violate his definitions. Or, more precisely, it does not violate his definitions unless it can be demonstrated that the formula p = forall(17, q) is not a class-sign. And, that can be demonstrated.

When Gödel states that the formula, p = forall(17, q), is a class-sign, this statement violates his definitions. On page 15, Gödel states that formula q has two free variables x and y. The term, forall(17, q), requires that we substitute for the variable x (Gödel number 17) a sign of the same type. The variable x in formula q is a sign of type 2 because its solution set is a set of integers, as opposed to an individual integer. Thus, only a variable that is a sign of type 2 may be substituted for variable x in formula q. But, for the formula, p = forall(17, q), to be a class-sign the substitute for variable x cannot be a sign of type 2. It must be a sign of type 1. Furthermore, the only member of the ambiguous category, sign of type 1, that can be substituted for variable x so that formula p becomes a class-sign is a number-sign. Gödel's statement that formula p is a class-sign violates the requirement that the only proper substitute for a variable is a variable that has the same sign of type. A similar argument demonstrates that Gödel's statement, "r is a primitive recursive class-sign,"[49] is also erroneous.

To complete the discussion of Formula (14) we now turn to the right side of the formula. The right side of Formula (14), subst(r, 17, number(x)), is derived from the fact that a portion of the left side of Formula (14) is equivalent to Formula (12), r = subst(q, 19, number(p)). The portion of the left side of Formula (14) that is equivalent to formula r is replaced by formula r and thus the right side of Formula (14) is produced.

Formula (14) might informally seem to be the immediate consequence of the preceding Formulas (9), (11), and (12). But, only if we are willing to accept some selective deletions and a violation of the requirement that the substitute for a variable must be a variable with the same sign of type.

Now, we come to one of the most significant errors in Gödel's formal proof. The core of the error consists of combining the left side of one formula with the right side of an entirely different formula in a manner that is completely unjustified. To delineate the nature of the error we must scrutinize the formation of both the left and right sides of Formulas (15) and (16). To do that we must begin by taking two steps. We must define the term insert, and then we must re-examine Formula (14).

On page 7, primitive recursive theorem I states, in part, “Every function (relation) that you get by inserting primitive recursive functions in the places of variables of other primitive recursive functions (relations) is itself primitive recursive. . . .”[50]

This theorem tells us that the insertion of a primitive recursive function in the place of a variable of another primitive recursive function, produces a primitive recursive function. It does not say it is invalid to insert primitive recursive functions for entire functions. In fact, the definition of insert(x, n, y) provided by concept 27 is very broad. It states that any item can be inserted in place of any other item. Concept 27 states the following:

“27. $\mathrm{Insert}(x, n, y) = \mathrm{argmin}\ z \leq (\mathrm{nthPrime}(\mathrm{length}(x) + \mathrm{length}(y)))^{x+y} \cdot \ \exists\, u, v \leq x \cdot$

$x = u \circ \mathrm{seq}(\mathrm{item}(n, x)) \circ v \ \wedge z = u \circ y \circ v \ \wedge n = \mathrm{length}(u) + 1$

You obtain insert(x, n, y) from x by inserting y instead of the nth item in the sequence x (provided that $0 < n \leq \mathrm{length}(x)$)” [51]

It turns out that there are two ways to correctly derive Formula (14) within the structure of the formal proof. The first way has already been described. It is to treat the formula subst(y, 19, number(y)) as a variable within a larger formula and substitute a variable of the same type for it, which is formula p, at least according to Gödel's accounting. The second way is to insert formula p for the entire formula subst(y, 19, number(y)), which is located on the left side of Formula (9). The important point is that Formula (14) cannot be obtained by inserting or substituting formula p for the formula y that occurs within formula subst(y, 19, number(y)), which is located on the left side of Formula (9).

We cannot derive Formula (14) by inserting or substituting formula p for formula y. But, what formula do we derive if we insert formula p for the formula y that occurs within formula subst(y, 19, number(y)), located on the left side of Formula (9)? We recall that formula p = forall(17, q). So we will insert forall(17, q) for formula y. We obtain the formula subst(forall(17, q), 19, number(forall(17, q))). Something very interesting will occur if we replace the term number(forall(17, q)) with the term number(p), which we can do since p = forall(17, q), but we should remember that this operation can generate a contradiction. We obtain the formula subst(forall(17, q), 19, number(p)). We recall from Formula (13) that, according to Gödel, subst(forall(17, q), 19, number(p)) = forall(17, r). Thus, when we insert formula p for the formula y that occurs within formula subst(y, 19, number(y)), we obtain forall(17,r). Of couse, we must recall that the validity of Formula (13) is extremely suspect.

Next, we should re-acquaint ourselves with Formulas (9) and (10) because formula p is inserted into these two formulas.

Formula (9)

~ proof For$_{\kappa}$(x, subst(y, 19, number(y))) ⇒ provable$_{\kappa}$(subst(q, 17 19, number(x) number(y)))

Formula(10)

proof For$_{\kappa}$(x, subst(y, 19, number(y))) ⇒ provable$_{\kappa}$(not(subst(q, 17 19, number(x) number(y))))

Directly following Formula (14) Gödel makes this key statement, "If we now insert p for y in (9) and (10), we get taking (13) and (14) into account:

Formula (15)

~proof For$_{k}$ (x, forall (17,r)) ⇒provable$_{k}$(subst(r, 17, number(x)))

Formula (16)

proofFor$_{k}$ (x, forall (17,r) ⇒ provable$_{k}$ (not(subst(r, 17, number(x))))" [52]

We can now appreciate Gödel's error. On the left sides of Formulas (15) and (16) Gödel has inserted formula p in the form of forall(17, q) for the formula y that occurs within the formula subst(y, 19, number(y)). Thus, through the use of the extremely suspect Formula (13), he changed subst(y, 19, number(y)) into forall(17, r). In a contradictory manner, the right sides of Formulas (15) and (16) are derived from inserting formula p in the form of forall(17, q) for the entire formula subst(y, 19, number(y)). Gödel produces each of the Formulas (15) and (16) by grafting together portions of two different formulas that should not be grafted together. This goes back to the nomenclature confusion mentioned before. The y in the term Q(x, y) does indeed refer to the entire formula subst(y, 19, number(y)), but confusing nomenclature isn't the proper grounding for deriving formulas for a mathematical proof.

Represented directly below are how Formulas (15) and (16) should appear when formula p in the form of forall(17, q) is inserted for the formula y that occurs in the formula subst(y, 19, number(y)), which occurs in both Formulas (9) and (10). This procedure generates the left side of Formulas (15) and (16).

Formula (15*)

$\sim$proof For$_k$ (x, forall (17,r)) $\Rightarrow$provable$_k$(subst(q, 17 19, number(x) number(forall (17, r))))

Formula (16*)

proof For$_k$ (x, forall (17,r)) $\Rightarrow$ provable$_k$ (not(subst(q, 17 19, number(x) number(forall(17, r)))))

If the forall(17, r) in the term number(forall(17, r) in the above formulas could legitamately be replaced with formula p we would have Formulas (15) and (16). This cannot be done because formula p has the free variable y and by Gödel's own definition forall(17, r) is a proposition formula, which has no free variables. Also, we have discussed before that Gödel's attempt in Formula (13) to equate a generalized version of formula p and forall(17, r) is invalid.

Represented directly below are how Formulas (15) and (16) should appear when formula p in the form of forall(17, q) is inserted for the entire formula subst(y, 19, number(y)), which occurs in both Formulas (9) and (10). It is interesting that Gödel chooses insertion instead of substitution at this juncture. The operation of substitution would be legitimate, according to Gödel, because as stated before formula p with the free variable y, and subst(y, 19, number(y)) with no free variables are both signs of type 1. Gödel may be tacitly supporting our analysis that formula p is a sign of type 2. Nonetheless, insertion produces the outcome Gödel desires for the right sides of the formulas. Of course, this procedure also generates changes in the left sides of Formulas (15) and (16). (Also, note that the right sides of both formulas have undergone the following changes: (1)the term number(forall(17, q) was replaced by the term number(p) because p = forall(17, q), although such a replacement can cause difficulties and (2) the term (q, 19, number(p)) was replaced with the term r because r = subst(q, 19, number(p)).

Formula (15**)

$\sim$proof $\mathrm{For}_k$ (x, forall (17, q)) $\Rightarrow$ $\mathrm{provable}_k$(subst (r, 17, number(x)))

Formula (16**)

proof $\mathrm{For}_k$ (x, forall (17, q)) $\Rightarrow$ $\mathrm{provable}_k$(not(subst(r, 17, number(x))))

Formula (15) is formed by combining the left side of Formula (15*) with the right side of Formula (15**). Formula (16) is formed in a similar manner; the left side of Formula (16*) is combined with the right side of Formula (16**). There is no justification for this combination. The formula p in the form of forall(17, q) is not equivalent to the formula forall(17, r). The formula forall(17, r) is a propositional-formula with no free variables. The formula p = forall(17, q) is a class-sign with one free variable y, at least according to Gödel. (Our analysis suggested formula p has two free variables.)

Formula (13) could be interpreted as evidence that formula p = forall(17, q) does equal forall(17, r) since, according to Gödel forall(17, r) =subst(p, 19, number(p)). But, as we have demonstrated the notion that the two formulas are equal is extremely suspect. Even the notion that subst(p, 19, number(p)) ⇒ forall(17, r) is suspect.

The formula subst(p, 19, number(p)) is important for another reason. If we incorrectly assume that the number-sign value (numeric value) of the Gödel number of the formula is number(p), instead of number(subst(p, 19, number(p))), we can use the formula subst(p, 19, number(p)) to derive Formula (14) through inserting formula subst(p, 19, number(p)) for the formula subst(y, 19, number(y)) in Formula (9). The end result is a situation where we generate contradictory statements. For instance, the unprovability of formula subst(p, 19, number(p)) implies the provability of formula subst(r, 17, number (x)). However, according to Formula (13) subst(p, 19, number(p)) = forall(17, r), therefore the unprovability of forall(17, r) implies the provability of subst(r, 17, number(x)). The contradiction is that forall(17, r) implies subst(r, 17, number(x)). This is another, perhaps more concise, route to justify Formulas (15) and (16), at least according to Gödel.

But, the number-sign value of the Gödel number of the formula subst(p, 19, number(p)) is not number(p). This is the same error we encountered in Formulas (9) and (10). The number-sign for the Gödel number of the formula subst(y, 19, number(y)) is incorrectly given as number(y), instead of the correct value number(subst(y, 19, number(y))). As you will recall the justification for this slight of hand was Formula (8.1), which states Q(x, y)$\Longleftrightarrow$ $\sim$(proof For$_{\kappa}$(x, subst (y, 19, number (y)))). The y in the term Q(x, y) does represent the formula subst(y, 19, number(y)), but it is incorrect to allow the number-sign of the Gödel number of the formula subst(y, 19, number(y)) to be designated number(y) instead of number(subst(y, 19, number(y))). It is incorrect because it confuses formula y with formula subst(y, 19, number(y)). The error cannot be corrected by claiming that the y in the term number(y) refers to the y in the term Q(x, y) because this conflation of quantities only produces more confusion.

Here is an example of the confusion. To which formula does the term Q(x, p) refer? Should it appear in this formulation Q(x, p)$\Longleftrightarrow$ $\sim$(proof For$_{\kappa}$(x, subst (p, 19, number (p)))) or in this formulation Q(x, p)$\Longleftrightarrow$ $\sim$(proof For$_{\kappa}$(x, forall(17, q)))? With regard to the latter formula, it is important to recall p = forall (17, q). To what value does the term number(p) refer? Number(p) could refer either to number(forall(17, q)) or number(subst(p, 19, number (p))) if we accept Gödel's confusing use of nomenclature.

It is this kind of confusion that runs through the entire the formal proof of Gödel's first incompleteness theorem.

## Conclusion

There are many, very serious flaws in Gödel's first incompleteness theorem. The theorem does not succeed in accomplishing its goal of demonstrating that in system P there is an arithmetical statement that is true, yet it is neither provable nor unprovable. There are at least three distinct methods that can be utilized to falsify Gödel's first incompleteness theorem.

The first method centers around the demonstration that axiom III,1 is invalid when constants (number-signs in Gödel's terminology) are substituted for variables. The constants under consideration are the natural numbers including zero. The axiom $(\forall\ v \cdot a) \Rightarrow$ subst $a(^{v}{}_{c})$ is correct when variables are substituted for other variables. It is also correct when the variables, which undergo substitution, represent entire formulas as long as certain precautions are taken. Gödel accurately advises that certain precautions must be taken when the variables, which undergo substitution, represent entire formulas, but he does not heed his own advise. The problem arises when natural numbers are substitutes for variables. This causes difficulties that cannot be surmounted.

The second method centers around the demonstration that Theorem V is seriously flawed. The terms number($x_1$) and number($x_n$) occurring in Formula (3) are not the same as the terms number($x_1$) and number($x_n$) occurring in Formula (4). The fact that Gödel represents them as equivalent terms is an error.  This error is utilized in a key portion of the formal proof of the first incompleteness theorem.  Without the use of this erroneous equivalency of terms, the formal proof is debilitated.

The third method centers around the demonstration that Formulas (15) and (16) are improperly formulated.  These formulas represent the culmination of the formal proof.  Both Formulas (15) and (16) are improperly constructed. They are each improperly constructed in a similar manner.  The left side of a certain formula is improperly combined with the right side of another formula.

We have tried to deal with all the troubling aspects of Gödel's paper. Any lingering questions should not deter us from arriving at a conclusion. Gödel's paper, On Formally Undecidable Propositions of Principia Mathematica and Related Systems I,  has a daunting title that is a harbinger of the complexity within, but despite its complexity, it is incorrect.

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## Endnotes

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