A Critique of Gödel's Theorem as It Is Presented by Ernest Nagel and James R. Newman in their Book *Gödel's Proof*

The book entitled *Gödel's Proof* was copyrighted in 1958 by the authors, and soon thereafter it was published by the New York University Press. Their book is dedicated to Bertrand Russell. The concluding paragraph of the book's introduction distills the essential purpose of the book. The authors write, "The details of Gödel's proofs in his epoch-making paper are too difficult to follow without considerable mathematical training. But the basic structure of his demonstrations and the core of his conclusions can be made intelligible to readers with very limited mathematical and logical preparation. To achieve such an understanding, the reader may find useful a brief account of certain relevant developments in the history of mathematics and of modern formal logic. The next four sections of this essay are devoted to this survey." The first six chapters of

Godel's Proof serve as an introduction to and general overview of the subject matter plus the topics of mapping in mathematics and proofs of consistency are also dealt with.

Chapter VII is entitled "Gödel's Proofs" and part C of that chapter bears the descriptive appellation, "The heart of Gödel's argument." The authors' description of Gödel's theorem is essentially the same as the one that would later be put forth by Douglas R. Hofstadter in his 1979 book *Gödel, Escher, Bach: An Eternal Golden Braid*. Both books make the same error, and it should be noted that this error does not appear in Gödel's original paper. It may be that the authors of both books were trying to overcome a very significant omission in the original paper.

On page 14 of Martin Hirzel's translation of Gödel's original paper, *On Formally Undecidable Propositions of <u>Principia Mathematica</u> and Related Systems*, Gödel writes, "We content ourselves with giving a sketchy outline of the proof for this theorem here, [Theorem V] since it does not offer any difficulties in principle and is rather cumbersome." Martin Hirzel's translation is dated November 27, 2000, and it is available at http://www.research.ibm.com/ h/hirzel/papers/canon00-goedel.pdf. Theorem V turns out to be at the heart of the formal proof of Theorem VI or Gödel's theorem. It is at the precise point where Theorem V is introduced into the formal proof of Theorem VI or Gödel's theorem that the authors of both books cease to follow the train of reasoning in Gödel's original paper and introduce their own interpretations.

Since Theorem V was never formally demonstrated in Gödel's original paper, the authors may have realized that it would overcome a very serious defect in the original paper if they could prove Theorem VI or Gödel's theorem without resorting to the introduction of Theorem V into the formal proof. Unfortunately, whether or not this was the motivation for the authors of both books, they failed in their attempts. Since Gödel's original paper only offers "a sketchy outline of the proof" of Theorem V and since Theorem V lies at the heart of the formal proof of Theorem VI or Gödel's Theorem, it is clear that Gödel's original paper does not contain a complete formal proof of Gödel's Theorem.

It is interesting to note that one of the two reasons Gödel gives for not producing a formal proof of Theorem V is that the proof is "cumbersome." This is typically a gambit employed in nontechnical explanations of difficult subject matter in mathematics and science. It is used in popular accounts of difficult subject matter, where it is assumed the average reader has no interest in the intricacies of the subject and only desires a simplified overview. Therefore, the avoidance of a "cumbersome" proof in such an abstruse paper that was written specifically for mathematicians who were attuned to this subject matter is inexplicable.

The specific error Nagel and Newman make occurs in their rendition of Gödel's theorem. It occurs in the term sub(*n,* 13, *n*), there the first "*n*" is, according to Nagel and Newman, the Gödel number of a certain specific formula; this is incorrect. It should represent the numeric value of (or the numeral for) the Gödel number *n* of a certain specific formula. But, their error is a bit more subtle than this description makes it seem because in a John Kerryesque manner before the authors allow the first "*n*" in the term sub(*n,* 13, *n*) to represent the incorrect value, the authors allow the first "*n*" to represent its correct value in an explanatory paragraph that precedes Gödel's theorem.

We pick up Nagel and Newman's argument on page 87, "... we have a new formula: '(*x*) ~ Dem (*x, z*)' ,which represents within the arithmetic the meta-mathematical statement: 'For every *x*, the sequence of formulas with Gödel number *x* is not a proof for the formula with Gödel number *z*....What Gödel showed is that a certain special case of this formula is not formally demonstrable. To construct this special case, begin with the formula displayed as line (1):

(1) (x) ~ Dem (*x*, sub(*y*, 13, *y*))."

The above formula is essentially the same as the formula that Hofstadter refers to as "*G's uncle*" on page 446 of his book. On the bottom of page 87 Nagel and Newman offer a concise explanation of their "pre-special case formula", which reads, "This formula belongs to the arithmetical calculus, but it represents a meta-mathematical statement. The question is, which one? The reader should first recall that the expression: 'sub(*y,* 13, *y*)' designates a number. This number is the Gödel number of the formula obtained from the formula with Gödel number *y*, by substituting for the variable with Gödel number 13 the numeral for *y*. It will then be evident that the formula of line (1) represents the meta-mathematical statement: ‘The formula with the Gödel number sub(*y,* 13, *y*) is not demonstrable.’”

The substitution operation denoted by Nagel and Newman with sub(*y,* 13, *y*) is essentially the same as Hofstadter’s **ARITHMOQUINE {a’’, a’}**. For Hofstadter’s operation the requirement was a formula with at least one free variable. For Nagel and Newman’s substitution operation the requirement is a formula, and it makes it easier to understand the substitution operation if the formula has a free variable and if that free variable is *y*.

For example, let’s chose the formula: 1 = *y* as the formula with Gödel number *y*. To find its Gödel number we must rewrite it in this form: s0 = *y*. This reads the number that is the immediate successor of zero equals *y*. The Gödel numbers of the elements of the equation are: s is 7, 0 is 6, = is 5 and *y* is 13. The Gödel number of the equation is: (2^7) (3^6) (5^5) (7^{13}). Therefore, the Gödel number of the equation represented by the letter (variable) *y* is: (2,825,283,544) (10^{10}). Now, we know the Gödel number of the equation: 1 = *y*. We also know that the Gödel number of the variable *y* in the equation is 13.

So to continue the substitution operation: we substitute the numeric value of the Gödel number (for the formula represented by the Gödel number *y*) for the variable *y* in the equation 1 = *y*. The variable *y* in the equation is represented by the Gödel number 13. The formula generated by the operation sub(*y,* 13, *y*) on the formula 1 = *y* is: s0 = ssss....(where there are (2,825,283,544) (10^{10}) s's.....ssss0. The Gödel number of the above formula would be an incredibly large number. And, also, the formula itself is false because (stated in everyday parlance) 1 isn't equal to (2,825,283,544) (10^{10}).

Now, we are ready to return to the "pre-special case formula" of Nagel and Newman which is: (*x*) ~ Dem(*x*, sub(*y,* 13, *y*) and we are about to come upon their error. The authors write on the top of page 89, "But, since the formula of line (1) belongs to the arithmetical calculus, it has a Gödel number that can actually be calculated. Suppose the number to be *n*. We now substitute for the variable with the Gödel number 13 (i.e., for the variable '*y*') in the formula of line (1) the numeral for *n.* A new formula is then obtained, which we shall call 'G' (after Gödel) and display under that label:

(G) (*x*) ~ Dem (*x*, sub(*n,* 13, *n*))."

And, there is their error. Instead of substituting the numeral for Gödel number *n* as they claimed they were going to do, they substituted the Gödel number *n* in the instance of the first "*n*" in formula (G). The above formula (G) should read: (*x*) ~ Dem (*x*, sub(the numeral for the Gödel number *n,* 13, *n*). The error is quite subtle.

What is the significance of their error? The authors go on to claim that the Gödel number of their formula (G) is: sub(*n,* 13, *n*), and again that is erroneous. But, it is again a subtle error because the Gödel number for the *corrected* formula (G) is: sub(*n,* 13, *n*). To make this point clearer, I should point out that it is assumed that the first "*n*" in the term sub(*n*, 13, *n*) refers to the formula with Gödel number *n* unless otherwise stated. That is why when I corrected (G) I explicitly stated the nature of the first "*n*."

Why is the Gödel number of the corrected version of (G) represented by this term: sub(*n*, 13, *n*)? This is so because the formula with Gödel number *n* is the formula: (*x*) ~Dem (*x,* sub(*y,* 13, *y*). And, if you substitute for the variable with the Gödel number 13 (which is the variable *y*), the numeral for (or the numeric value of) the

Gödel number *n*, you will generate the corrected formula (G).

But, if one is mistakenly convinced that the authors' formula denoted by (G) is legitimate, then an undecidable proposition is the end result, and this occurs whenever this formula is interpreted into a verbal statement. The authors' formula (G): (x) ~ Dem (*x,* sub(*n,* 13, *n*) means: For all *x*'s where *x* represents a sequence of formulas that form a proof then it follows that these various sets of sequences of formulas do not represent or demonstrate various sets of sequences of formula. The formula denies the existence of its self-evident content.

The interpretation of the corrected version of (G) is: For all *x*'s where *x* represents a sequence of formulas that form a proof then it follows that these proofs cannot demonstrate the non-formula sub(the numeric value of Gödel number *n*, 13, *n*) because it merely represents some number. It is like saying you can't prove "3 + 3."

An argument could be made that the authors' formula denoted by (G) is the same as my corrected version of their formula, or it is only a slightly more ambiguous version of my corrected version. This argument would be supported by what the authors write on page 89, "We now substitute for the variable with Gödel number 13 (i.e., for the variable '*y*') in the formula of line (1) the numeral for *n*." But, that argument cannot be sustained because it would rule out the fundamental conclusion drawn by the authors, "...the formula (G) is the mirror image *within* the arithmetical calculus of the meta-mathematical statement: 'The formula with the Gödel number sub(*n,* 13, *n*) is not demonstrable'....In a sense, therefore, this arithmetical formula (G) can be construed as asserting of itself that it is not demonstrable."

There is only one way the last term in the formula (G) i.e., sub(*n,* 13, *n*) could refer to the formula(G) i.e., the formula with the Gödel number sub(*n,* 13, *n*) and that is if the first "*n*" represents the formula with Gödel number *n*. And, as I have shown, that interpretation represents an incorrect interpretation of what the first "*n*" represents in the term sub(*n,* 13, *n*) <u>as the term is employed in formula (G).</u> But, it does not represent an incorrect interpretation of what the first "*n*" represents in the term sub(*n,* 13, *n*) when term is being employed to represent the Gödel number of the formula (G). And, that is the source of the confusion.